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Stability analysis of stochastic delay differential equations with Lévy noise ☆

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ABSTRACT

This paper is concerned with *p*th moment exponential stability problem for a class of stochastic delay differential equations driven by Lévy processes. Several new stability theorems are obtained by developing a method—proof by contradiction. Moreover, the results are applied to investigate the *p*th moment exponential stability of stochastic neural networks with Lévy noise. In particular, the time-varying delay in our results is not required to be differentiable, even not continuous. The obtained results improve greatly some previous works given in the literature. In particular, our method can easily correct the incorrect proofs appeared in two recent papers. Finally, two examples are provided to show the effectiveness of the theoretical results.

1. Introduction

In recent decades, stochastic differential equations (SDEs) have been widely studied since they have been successfully applied in many fields such as physical, biological, engineering, medical, social sciences, economics, finance and so on. One of the best important works in this research field is to discuss the stability of such systems [1–4]. For example, Dragan and Mukaidani in [5] studied the stabilization problem for singularly perturbed linear stochastic systems with state-multiplicative white-noise and Markovian jumping parameters. By introducing the generalized local Lipschitz condition and one-side linear growth condition, Zhao and Deng in [6] discussed the stochastic stabilization and destabilization problem of stochastic systems. In [7], Ito and Nishimura developed tools to investigate the stability and robustness of cascaded nonlinear stochastic systems based on Lyapunov functions. Mateos-Núñez and Cortés in [8] discussed the *p*th moment noise-to-state stability of SDEs with persistent noise. For more results we refer the reader to the books [9,10] and the survey paper [11]. Indeed, for a given SDE, it is very important to determine whether the system is stable.

On the other hand, time delays are often encountered in some real engineering problems, and the existence of time delays may cause oscillation or instability in SDEs, which are harmful to the applications of SDEs. Therefore, time delays must be taken into account when studying the stability of SDEs. Usually, SDEs with time delays are called stochastic delay differential equations (SDDEs). The stability of SDDEs has been widely studied in the literature. For instance, Kang et al. in [12] studied the stability of hybrid SDDEs with an asynchronous switching controller. Hang and Mao in [13] discussed the mean-square exponential stability and almost surely exponential stability of neutral SDDEs by using the linear matrix inequality approach. Hu et al. in [14] investigated the robust stability and boundedness of nonlinear hybrid SDDEs without using the linear growth condition. For more results we refer the reader to [9,10,15–21] and the references therein. As we know, the stability criteria on SDDEs are mainly including stochastic stability, asymptotical stability, almost sure stability and exponential stability. Generally speaking, the exponential stability is recognized to the best stability behavior since it not only ensures the stability of the system but also gives an explicit convergence rate. However, it is difficult to realize in some real systems. Therefore, it is interesting and challenging to study the exponential stability of SDDEs.

We now introduce some existing works on the exponential stability of SDDEs. It is known that the powerful technique used in the study of exponential stability of SDDEs is based on a stochastic version of the Lyapunov direct method [9,10]. The key of this method is to find effectively Lyapunov functionals or Lyapunov functions V combined with the Itô operator $\mathcal{L}V$. The classical method requires that the Itô operator $\mathcal{L}V$ of SDEs is negative, i.e., $\mathcal{L}V < 0$ (e.g., see the book [9] and the references therein). For the case of SDDEs, there are more restriction conditions: The conditions $\mathcal{L}V(x, y, t) \le -\lambda_1|x|^p + \lambda_2|y|^p$, $\lambda_1 > \frac{\lambda_2}{1-\delta} > 0$ and $\dot{\tau}(t) \le \delta < 1$ are required when the delay $\tau(t)$ is time-varying (e.g., see [10,17,21] and the references therein). However, these conditions are sometimes too restrictive to fail for some

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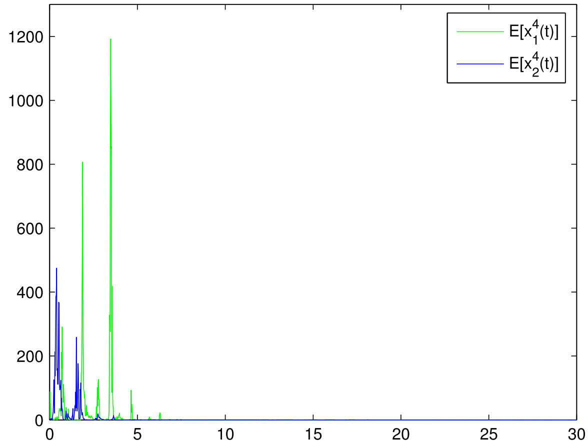

Fig. 1. 4th moment of the solution.

real models. In fact, the delays can occur in an irregular fashion due to the complexity of real systems, which leads to their non-differentiability (e.g., the signal has to stop propagating for a time when it encounters an obstacle, but when the obstacle moves away it will continue to propagate again). Now, let us consider the following simple linear SDDE: $dx(t) = [-3x(t) + x(t - \tau(t))]dt + 1.9x(t - \tau(t))dB(t)$, where $\tau(t) = 0.6|\sin t| + 0.5$. Obviously, the stability criteria in the previous literature (e.g. [10,17,21]) fail in this model since $\tau(t) = 0.6|\sin t| + 0.5$ is not differentiable. In fact, even if we allow $\tau(t)$ to be differentiable and satisfy $\dot{\tau}(t) \le \delta < 1$, the criteria in [10,17,21] are still very conservative. For example, taking $\tau(t) = 0.6\sin t + 0.5$, it is clear that $\delta = 0.6$. Then, we take $V(x,t) = |x|^2$ and compute $\mathcal{L}V(x,y,t) = 2x^T(-3x + y) + y^2 \le -6|x|^2 + 2x^Ty + 3.61|y|^2 \le -5|x|^2 + 4.61|y|^2$, which yields $\lambda_1 = 5$ and $\lambda_2 = 4.61$. However, $\lambda_1 = 5 < \frac{\lambda_2}{1-\delta} = \frac{4.61}{1-0.6} = 11.525$, which implies that the criteria in [10,17,21] fail in this case. In fact, this model is indeed 4th moment exponentially stable, and Fig. 1 further shows the stability. Naturally, an interesting question arises: under what conditions can SDDEs realize the pth moment exponential stability when $\tau(t)$ is not differentiable? To the best of our knowledge, up to now, there has been few results on this problem.

Motivated by the above discussion, in this paper we are interested in the pth moment exponential stability of SDDEs. Our aim is to establish some new stability results by developing a method—proof by contradiction. To make the results allow more general real models, we also consider the effect of Lévy noise. In fact, many real random events like the fluctuations in the financial markets and large disasters in life sciences are not described by the Brown motion. It is recognized that Lévy noise can described these complex random evens but it makes the analysis more difficult owing to the discontinuity of its sample paths [18–22]. In this paper, we attempt to overcome this difficulty and obtain several novel stability theorems. Compared with the previous results, the contributions of this paper lie in three aspects as follows.

(1) Without additional restrictive differentiable conditions on the time-varying delay, the pth moment exponential stability problem for a class of SDEs with time-varying delays has been studied. The time-varying delay in the previous literature is required to be differentiable and the bound of its derivative is also required to be less than 1 (e.g., see [10,15,17,21,23,24]).

(2) We develop a method—proof by contradiction, which is different from those used in [10,15,17,21,23–25]. Moreover, the Gronwall inequality method used in [17,25] is wrong (see Remark 4 in detail). In fact, our method can easily correct the incorrect proofs appeared in [17,25].

(3) Lévy noises have been considered in this paper. In particular, we consider the effects of both "small jump" and "large jump". However, Lévy noises were ignored in [10,15,17,23–25].

The remainder of this paper is organized as follows. In Section 2, we introduce the model, notations and some necessary assumptions. In Section 3, we establish several new stability theorems by developing a novel method and apply our results to stochastic neural networks. In Section 4, we use two examples to show the effectiveness of the obtained results. Finally, in Section 5, we conclude this paper with some general remarks.

2. Model, notations and assumptions

Notations. Throughout this paper, we will use the following notations. $\mathbb{R}^d$ denotes the d-dimensional Euclidean space, and $|x|$ denotes the Euclidean norm of a vector x. Take $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}_+ = [0, +\infty)$. The superscript "$\mathbb{T}$" denotes the transpose of a matrix or vector, and trace$(\cdot)$ denotes the trace of the corresponding matrix. For any matrix A, $\lambda_{\max}(A)$ (respectively, $\lambda_{\min}(A)$) denotes the largest (respectively, smallest) eigenvalue of A. $a \vee b = \max\{a, b\}$.

Let $B = (B(t) = (B_1(t), \ldots, B_m(t))^T, t \ge 0)$ be an m-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\ge0}$. Also, let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^d)$ denote the family of continuous functions ϕ from $[-\tau, 0]$ to $\mathbb{R}^d$ with the uniform norm $\|\phi\|_\tau = \sup_{-\tau\le\theta\le0}|\phi(\theta)|$. Let $p > 0$ and denote by $L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^d)$ the family of all $\mathcal{F}_0$ measurable, $C([-\tau, 0]; \mathbb{R}^d)$-valued stochastic variables $\phi = \{\phi(\theta) : -\tau \le \theta \le 0\}$ such that $\mathbb{E}\,\|\phi\|^p_\tau < \infty$, where $\mathbb{E}$ stands for the correspondent expectation operator with respect to the given probability measure $\mathbb{P}$.

In this paper, we will discuss the following stochastic delay differentiable equation driven by Lévy processes:

$$\begin{aligned} &dx(t)\\ &= f(x(t), x(t-\tau(t)), t)dt\\ &\quad + g(x(t), x(t-\tau(t)), t)dB(t)\\ &\quad + \int_{|z|<c} H_1(x(t^-), x(t-\tau(t))^-, t, z)\tilde{N}(dt, dz)\\ &\quad + \int_{|z|\ge c} H_2(x(t^-), x(t-\tau(t))^-, t, z)N(dt, dz), \end{aligned} \tag{1}$$

where the initial data $x_0 = \xi = \{\xi(\theta), -\tau \le \theta \le 0\} \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^d)$, $x(t^-) = \lim_{s\uparrow t} x(s)$, the mappings $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$, $g : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times m}$ denotes the space of all real-valued $d \times m$ matrices, $H_i : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d (i = 1, 2)$, and the constant $c \in (0, \infty]$ is the maximum allowable jump size. $\tau(t)$ is a time-varying delay and satisfies $0 \le \tau(t) \le \tau$. N is a Poisson random measure defined on $\mathbb{R}_+ \times (\mathbb{R}^d - \{0\})$ with compensator $\tilde{N}$ and intensity measure ν. We assume that N is independent of B and ν is a Lévy measure such that $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$ and $\int_{\mathbb{R}^d-\{0\}}(|z|^p \bigwedge 1)\nu(dz) < \infty$. Usually, the pair (B, N) is called a *Lévy noise*, $\int_{|z|<c} H_1(x(t^-), x(t-\tau(t))^-, t, z)\tilde{N}(dt, dz)$ is called "small jump" and $\int_{|z|\ge c} H_2(x(t^-), x(t-\tau(t))^-, t, z)N(dt, dz)$ is called "large jump".

To ensure that system (1) has a unique solution, throughout this paper we assume that the functions f, g, H_1 and H_2 satisfy the local Lipschitz condition, $f(0, 0, t) = 0$, $g(0, 0, t) = 0$, $H_1(0, 0, t, z) = 0$ for all $t \in \mathbb{R}_+$ and $|z| < c$, and $H_2(0, 0, t, z) = 0$ for all $t \in \mathbb{R}_+$ and $|z| \ge c$. Under these conditions, it follows from [20] and [22] that (1) has a unique solution $x(t; \xi)$ on $t \ge t_0 - \tau$. In particular, $x(t; \xi) = 0$ for all $t \ge t_0 - \tau$ corresponding to the initial data $\xi \equiv 0$, which is often called the *trivial solution*.