Systems & Control Letters 118 (2018) 94–100

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

On controllability of time-varying linear population systems with parameters in unbounded sets ☆

Wei Zhang, Jr-Shin Li *

Department of Electrical and Systems Engineering, Washington University in St. Louis, St. Louis, MO 63130, USA

ARTICLE INFO

Article history:
Received 5 December 2017
Received in revised form 19 May 2018
Accepted 23 May 2018

Keywords:
Ensemble control
Population systems
Parameter-dependent systems
Singular systems
Hilbert–Schmidt operators

ABSTRACT

The ability to finely manipulate a population of structurally identical dynamical systems is important for various emerging applications across a broad spectrum of fields in science and engineering. Robust control of such an ensemble is fundamentally challenging, because individual systems in the ensemble obey the same dynamical law but have different system parameter values, and only a broadcast control can be applied to the entire ensemble. In this paper, we establish controllability conditions for the finite-dimensional time-varying linear ensemble system in a Hilbert space, whose parameters are in an unbounded set. This work extends our previous study in one-parameter families of linear ensemble systems with the parameter lying in a one-dimensional compact set.

1. Introduction

The problem of controlling populations of structurally identical dynamical systems with slightly different natural dynamics has attracted extensive attention in recent years. Notable examples arose from the domains of biology [1,2], neuroscience [3,4], and quantum physics [5–8], where targeted coordination of ensemble systems, e.g., cells, neurons, and spins, respectively, is an essential task to enable cutting-edge applications in these areas [9–12]. Numerous studies from both theoretical and computational aspects have been proposed for the control of ensemble systems, including the analysis of ensemble controllability [13–19] and ensemble observability [20,21], and the development of computational methods for optimal ensemble control synthesis [7,12,22–24].

In this paper, we extend our previous work on the control of one-parameter families of time-varying linear ensemble systems with a bounded parameter set [14] and consider linear ensemble systems characterized by multiple parameters over an unbounded set. Specifically, we analyze controllability of such ensemble systems in a Hilbert space setting and identify the types of system (i.e., drift and control) dynamics that will lead to the L_2-ensemble controllability, i.e., controllability defined with respect to the L_2-topology. In the next section, we will briefly review the previous results on the control of linear ensemble systems, and in Section 3 we will construct the controllability conditions on the system and control matrices for the linear ensemble system to be L_2-ensemble controllable. In Section 4, we present examples to describe and support the developed theoretical results.

2. Preliminaries to ensemble control of linear systems

Control of one-parameter time-varying linear ensemble systems in Hilbert spaces has been previously investigated [14]. In this section, we provide a brief review of the relevant previous results for controlling linear ensemble systems, and, meanwhile, define the necessary mathematical tools and notations.

2.1. Controllability of one-parameter families of linear ensemble systems in Hilbert spaces

Consider a parameterized family of finite-dimensional time-varying linear systems indexed by a parameter β varying over a compact set $K \subset \mathbb{R}$, given by

$$\frac{d}{dt}X(t,\beta) = A(t,\beta)X(t,\beta) + B(t,\beta)u(t), \tag{1}$$

where $X(t,\cdot) : K \to \mathbb{C}^n$ is the state, in which $K = [\beta_1, \beta_2] \subset \mathbb{R}$, and $u \in L_2^m[0,T]$, an m-tuple of L_2-functions, is the control; the matrices $A(t,\beta) \in \mathbb{C}^{n\times n}$ and $B(t,\beta) \in \mathbb{C}^{n\times m}$ have elements that are complex-valued L_∞- and L_2-functions, respectively, defined on the compact set $D = [0,T] \times K \subset \mathbb{R}^2$, and are denoted $A \in L_\infty^{n\times n}(D)$ and $B \in L_2^{n\times m}(D)$.

☆ This work was supported in part by the National Science Foundation under the award CMMI-1462796 and by the Air Force Office of Scientific Research under the award FA9550-17-1-0166.
* Corresponding author.
E-mail addresses: wei.zhang@wustl.edu (W. Zhang), jsli@wustl.edu (J.-S. Li).

https://doi.org/10.1016/j.sysconle.2018.05.014
0167-6911/

Definition 1 (L_2*-ensemble Controllability*)**.** An ensemble system as in (1) is said to be L_2*-ensemble controllable* on the function space $L_2^n(K)$ if for all $X_0, X_F \in L_2^n(K)$ and for any $\varepsilon > 0$, there exists a finite time $T > 0$ and a piecewise-continuous open-loop control $u : [0, T] \to \mathbb{C}^m$ such that starting from $X_0 \in L_2(K)$, the final state at time T, $X_T \in L_2^n(K)$, satisfies $\|X_T - X_F\|_2 < \varepsilon$ [13].

Note that the final time T in Definition 1 may depend on ε. Furthermore, ensemble controllability can also be defined with respect to other L_p-norms for $1 \le p \le \infty$, namely, the system is L_p-ensemble controllable if $(\int_K \|X(T, \beta) - X_F(\beta)\|^p \, d\beta)^{\frac{1}{p}} < \varepsilon$. Note that the notion of ensemble controllability is consistent with that of approximate controllability for infinite-dimensional control systems, such as systems described by partial differential equations or by delay differential equations [25]. In this paper, our focus is on the investigation of L_2-ensemble controllability of the system in (1).

It has been shown in [14] that the family as in (1) is L_2-ensemble controllable on the function space $L_2^n(K)$ if and only if for any given initial and final states, $X_0, X_F \in L_2^n(K)$ at $t = 0$ and $t = T < \infty$, respectively, with $\xi = \Phi(0, T, \beta)X_F - X_0$, the conditions

$$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{|\langle \xi, \nu_n \rangle|^2}{\sigma_n^2} < \infty, \qquad \text{(ii)} \quad \xi \in \overline{\mathcal{R}(L)}, \tag{2}$$

hold, where $\Phi(t, 0, \beta)$ is the transition matrix of the homogeneous system $\frac{d}{dt}X(t, \beta) = A(t, \beta)X(t, \beta)$ and L is the input-to-state operator associated with the system in (1), defined by

$$(Lu)(\beta) = \int_0^T \Phi(0, \tau, \beta)B(\tau, \beta)u(\tau)d\tau, \tag{3}$$

and where (σ_n, μ_n, ν_n) is a singular system of L (see Appendix A.1) and $\overline{\mathcal{R}(L)}$ denotes the closure of the range space of L.

3. Control of multi-parameter linear ensemble systems

The previous results described in Section 2 are concerned with the time-varying linear ensemble system characterized by a single bounded parameter. Here, we generalize these previous findings to establish conditions on the system and control matrices, $A(t, \beta)$ and $B(t, \beta)$, respectively, under which the system governed by multiple unbounded parameters is L_2-ensemble controllable. Specifically, these conditions are derived for the sake of guaranteeing compactness of the input-to-state operator L that governs the ensemble dynamics.

Specifically, we study the following time-varying linear ensemble system of d parameters of the form,

$$\Sigma : \begin{cases} \dfrac{d}{dt}X(t, \beta) = A(t, \beta)X(t, \beta) + B(t, \beta)u(t), \\ \beta \in \Omega \subset \mathbb{R}^d, \quad d > 1, \end{cases} \tag{4}$$

where $X : D \to \mathbb{C}^n$, in which $D = [0, T] \times \Omega \subset \mathbb{R}^{d+1}$, $u(t) \in \mathbb{C}^m$, $A : D \to \mathbb{C}^{n\times n}$, and $B : D \to \mathbb{C}^{n\times m}$. We first observe that the conditions in (2) were developed based upon the boundedness of the system matrix $A(t, \beta)$ over a compact set, so that the transition matrix $\Phi(t, 0, \beta)$ is the limiting matrix-valued function to the Peano–Baker series associated with $A(t, \beta)$. It follows that essential boundedness of $A(t, \beta)$ may also be sufficient to guarantee the boundedness of the transition matrix when the parameter set $\Omega \subset \mathbb{R}^d$ is unbounded. The following mathematical tools are essential to facilitate our developments of ensemble controllability conditions.

3.1. Preliminaries

Recall that $\mathbb{C}^{n\times m}$ equipped with the Frobenius norm $\|\cdot\|_F$, where $\|M\|_F = \sqrt{trM^\dagger M} = \sqrt{\sum_{i=1}^n \sum_{j=1}^m |M_{ij}|^2}$ for $M \in \mathbb{C}^{n\times m}$, is a Banach space. Let (X, μ) be a measurable subspace of $\mathbb{R}^k$ with the Lebesgue measure μ and $A : X \to \mathbb{C}^{n\times m}$ be a matrix-valued function. Then, we can define the Bochner integral in terms of simple functions, which are finite linear combinations of characteristic functions of measurable sets.

Definition 2 (*Bochner Integrability*)**.** A measurable function $A : X \to \mathbb{C}^{n\times m}$ is *Bochner integrable* if there exists a sequence of simple functions $A_1, A_2, \ldots, A_n, \ldots$ such that $\lim_{n\to\infty} \int_X \|A - A_n\|_F \, d\mu = 0$.

It may be intractable to examine the Bochner integrability through Definition 2, and thus an alternative characterization of Bochner integrable functions is given below.

Lemma 1. *Let $A : X \to \mathbb{C}^{n\times m}$ be a matrix-valued function, where $X \subset \mathbb{R}^k$, then A is Bochner integrable if and only if $\int_X \|A\|_F \, d\mu < \infty$* [26].

Following Lemma 1, we define the *Bochner space* of matrix-valued L_p-functions, $L_p^{n\times m}(X, \mu) = \{A : X \to \mathbb{C}^{n\times m} \mid \|A\|_p < \infty\}$, where $\|A\|_p = \left(\int_X \|A\|_F^p \, d\mu\right)^{\frac{1}{p}}$ for $1 \le p < \infty$. For $p = \infty$, we have $L_\infty^{n\times m}(X, \mu) = \{A : X \to \mathbb{C}^{n\times m} \mid \|A\|_\infty < \infty\}$, where $\|A\|_\infty = \text{esssup}_{x\in X}\|A(x)\|_F$.

3.2. Characterization of ensemble controllability in Hilbert spaces

Using the terminology of Bochner integral introduced above, we construct the conditions that render the boundedness of the transition matrix associated with the system Σ in (4).

Proposition 1. *Consider the ensemble system Σ as in (4), if $A \in L_\infty^{n\times n}(D)$, then the transition matrix $\Phi \in L_\infty^{n\times n}(D)$, where $D = [0, T] \times \Omega \subset \mathbb{R}^{d+1}$ and $\beta \in \Omega \subset \mathbb{R}^d$.*

Proof. The transition matrix of the system in (4) can be represented by the Peano–Baker series,

$$\Phi(t, 0, \beta) = I + \int_0^t A(\sigma_1, \beta)d\sigma_1 + \int_0^t A(\sigma_1, \beta) \times \int_0^{\sigma_1} A(\sigma_2, \beta)d\sigma_2 \, d\sigma_1 + \cdots .$$

For each fixed $(t, \beta) \in D$, the triangle inequality of $\|\cdot\|_F$ gives $\|\Phi(t, 0, \beta)\|_F \le \sqrt{n} + \left\| \int_0^t A(\sigma_1, \beta)d\sigma_1 \right\|_F + \left\| \int_0^t \int_0^{\sigma_1} A(\sigma_1, \beta)A(\sigma_2, \beta)d\sigma_2 d\sigma_1 \right\|_F + \cdots \le n + n\int_0^t \|A(\sigma_1, \beta)\|_F d\sigma_1 + n\int_0^t \int_0^{\sigma_1} \|A(\sigma_1, \beta)\|_F \|A(\sigma_2, \beta)\|_F d\sigma_2 d\sigma_1 + \cdots$, which converges to $n \exp\left[\int_0^t \|A(\tau, \beta)\|_F d\tau\right]$, where the second inequality follows from the relation of the L_2- and L_1-vector norms, e.g., $\| \int_0^t A(\sigma_1, \beta)d\sigma_1\|_F \le \int_0^t \sum_{i,j}|A_{ij}(\sigma_1, \beta)|d\sigma_1 \le \int_0^t n\sqrt{\sum_{i,j}|A_{ij}(\sigma_1, \beta)|^2}d\sigma_1 = n\int_0^t \left\|A(\sigma_1, \beta)\right\|_F d\sigma_1$, with A_{ij} denoting the *ijth* element of A, and the sub-multiplicativity of $\|\cdot\|_F$ [27]. Let S be the set defined by $S = \{(t, \beta) \in \mathbb{R}^{d+1} : \|A\|_F \le \|A\|_\infty\}$. Because $A \in L_\infty^{n\times n}(D)$, $D - S$, the complement of S in D, has Lebesgue measure 0. Then, for each $(t, \beta) \in S$, $\|\Phi(t, 0, \beta)\|_F \le ne^{\|A\|_\infty T} < \infty$ and hence $\Phi \in L_\infty^{n\times n}(D)$. □

Remark 1. It follows from Proposition 1 that $\Phi(0, t, \beta) = \Phi^{-1}(t, 0, \beta) \in L_\infty^{n\times n}(D)$. This can be seen from the fact that $\Phi(0, t, \beta)$ is the transition matrix associated with the system $\frac{d}{dt}X(t, \beta) = -A'(t, \beta)X(t, \beta)$, where $-A'(t, \beta)$ is essentially bounded if $A(t, \beta)$ is.

In the following, we consider the ensemble system in (4) and show different scenarios for the system matrices $A(t, \beta)$ and control matrices $B(t, \beta)$ that lead to L_2-ensemble controllability.