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Integral versions of input-to-state stability for dual-rate nonlinear sampled-data systems☆

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ABSTRACT

This paper presents versions of integral input-to-state stability and integral input-to-integral-state stability for nonlinear sampled-data systems, under the low measurement rate constraint. In particular, we compensate the lack of measurements using an estimator approximately reconstructing the current state. Interestingly, under certain checkable conditions, we establish that a controller that semiglobally practically integral input-to-(integral-)state stabilizes an approximate discrete-time model of a *single-rate* nonlinear sampled-data system, also stabilizes the exact discrete-time model of the nonlinear sampled-data system in the same sense implemented in a *dual-rate* setting. Numerical simulations are given to illustrate the effectiveness of our results.

1. Introduction

Sampled-data systems consist of a continuous-time plant/process controlled by a digital controller interfaced through analogue-to-discrete (A/D) and discrete-to-analogue (D/A) converters. When the plant is nonlinear, analysis and design of sampled-data systems present significant challenges and have attracted significant research attention; e.g., [1–5].

Sampled-data systems can be designed by *emulation* [6–8] or in *discrete-time* [9–11]. In the emulation approach, a continuous-time controller is first designed for the continuous-time plant and then the controller is discretized using exiting techniques for numerical integration. In the discrete-time approach, on the other hand, first the plant model is discretized and then a controller is designed for the resulting discrete-time plant.

In general, the emulation approach presents the disadvantage that requires high sampling rates in order to recover the properties of the analog design including stability and desired performance. The discrete-time approach can usually offer similar results at lower sampling rates compared to emulation-based controllers. Unfortunately, however, finding the exact discrete-time model of a continuous-time plant requires solving an initial value problem whose solution does not exist in closed-form for any practical nonlinear plant model. A remedy for this issue is to conclude stability properties and/or system performance of the exact discrete-time model from an approximate model of the system. This treatment has received considerable attention over the last two decades; e.g., [12–15]. It should be noted that there have been other approaches such as [1,4], where smoothness conditions on the model describing the system are required.

In the discrete-time approach, it is usually assumed that all signals in the loop are sampled regularly at the same sampling rate. Practically speaking, however, the use of different sampling rates for inputs and outputs is highly desirable or even mandatory. A dual-rate sampled-data stabilization problem is studied in [16], where low measurement rates are considered and a multi-rate controller, that approximates state trajectories between samples, is proposed. In [17], the work [16] is extended to include the effect of disturbances, using the concept of input-to-state stability (ISS) [18]. Furthermore, extensions to multi-rate output feedback control problem are reported in [19,3].

Major outcome of this body of work is the understanding of under what conditions stability properties of a digital design based on an approximate discrete-time model of a sampled-data system, are also true for the exact (not available to the designer) discrete-time model of the system. Particularly relevant to this work is the preservation of the important notions of ISS, and integral input-to-state stability (iISS) [20]. Both concepts are of fundamental importance in control theory. Informally, ISS and iISS capture the notion that the system state remains small, regarding

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the initial conditions, provided that the input (ISS) or integral of the input (iISS) is small. In [13], the authors consider single-rate sampled-data systems and provide sufficient conditions under which a discrete-time controller that input-to-state stabilizes an *approximate* discrete-time model of a nonlinear plant with disturbances also input-to-state stabilizes the *exact* discrete-time plant model. In [17], the authors consider low measurement frequency restriction and generalize the results in [13] to the dual-rate case. Integral versions of ISS, called semiglobal practical integral input-to-state stability (SP-iISS) and semiglobal practical integral input-to-integral-state stability (SP-iIiSS), of single-rate sampled-data systems via approximate models are considered in [21].

In this paper, our main interest is the study of dual rate iISS nonlinear sampled-data systems. As we will see later, this is a source of difficulties with which this work is concerned. Specially, we use an approximate discrete-time model of the plant to estimate inter-sampled values used by the controller working at the higher rate. Then we show SP-iI(i)SS for the exact discrete-time model of the dual-rate sampled-data based on checkable sufficient conditions obtained from a *single-rate* approximate model of the sampled-data system. As a matter of fact, analysis of a single-rate sampled-data system is, in general, much easier than that for its multi-rate counterpart. Therefore, our results may be preferable to the designer as we conclude stability of a dual-rate system from a single-rate model of the system. Eventually, we illustrate the effectiveness of our results via a numerical example.

The rest of this paper is organized as follows: Section 2 defines the notation used throughout the paper. Problem statement is given in Section 3 and the main results are provided in Section 4. An illustrative example is given in Section 5. Finally, concluding remarks are provided as the last section.

2. Notation

Throughout the paper, $\mathbb{R}_{\geq 0}(\mathbb{R}_{>0})$ and $\mathbb{Z}_{\geq 0}(\mathbb{Z}_{>0})$ are the nonnegative (positive) real and nonnegative (positive) integer numbers, respectively. We denote the standard Euclidean norm by $|\cdot|$. A function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is positive definite ($\rho \in \mathcal{PD}$) if it is continuous, zero at zero and positive elsewhere. A positive definite function α is of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is strictly increasing. It is of class $\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if $\alpha \in \mathcal{K}$ and also $\alpha(s) \to +\infty$ if $s \to +\infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ ($\beta \in \mathcal{KL}$), if for each $s \geq 0$, $\beta(\cdot, s) \in \mathcal{K}$, and for each $r \geq 0$, $\beta(r, \cdot)$ is decreasing and $\lim_{s\to+\infty}\beta(r, s) \to 0$. For a given function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$, $w_T[k]$ represents the restriction of the function $w(\cdot)$ to the interval $[kT, (k+1)T]$, where $k \in \mathbb{Z}_{\geq 0}$ and $T > 0$. Given $\gamma \in \mathcal{K}$ and a measurable function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$, we define $|w|_\gamma := \int_0^{+\infty} \gamma(|w(s)|)ds$. We denote the set of all functions with $|w|_\gamma < +\infty$ by $\mathcal{L}_\gamma$. Also, if $|w|_\gamma < r$ for some $r \in \mathbb{R}_{>0}$, we write $w \in \mathcal{L}_\gamma(r)$. Similarly, let $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ be a measurable function such that $|w|_\infty := \sup_{t\geq 0}|w(t)| < +\infty$. We denote the set of all such functions by $\mathcal{L}_\infty$. Also, if $|w|_\infty < r$ for some $r \in \mathbb{R}_{>0}$, we write $w \in \mathcal{L}_\infty(r)$.

3. Problem statement

Consider the following plant model

$$\dot{x}(t) = f(x(t), u(t), w(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $w(t) \in \mathbb{R}^p$ are the state, the control input and the disturbance input, respectively. The function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is locally Lipschitz and $f(0, 0, 0) = 0$. Moreover, we assume that $w(.)$ is measurable and locally essentially bounded.

We consider the dual-rate scenario where the plant (1) is connected to a computer control via a zero-order hold (ZOH) and a sampler. The control signal is held constant during the sampling intervals, that is, $u(t) = u(kT)$ for all $t \in [kT, (k+1)T)$, $k \in \mathbb{Z}_{\geq 0}$, where $T > 0$ is the sampling period. For simplicity, we abuse our notation and write $u(k) := u(kT)$ and $x(k) := x(kT)$. By T and T_s, we, respectively, denote the sampling periods of the ZOH and the sampling devices. We assume that $T_s = \ell T$ for some integer $\ell \geq 1$. Note that $\ell = 1$ implies the case of single-rate fashion which does not often hold in practice. The exact discrete-time plant model can be obtained from (1) as follows

$$\begin{aligned} x(k+1) &= x(k) + \int_{kT}^{(k+1)T} f(x(\tau), u(k), w(\tau))d\tau \\ &=: F_T^e(x(k), u(k), w_T[k]). \end{aligned} \tag{2}$$

Unfortunately, this requires solving the initial value problem (2) which does not have a closed-form solution, in most cases of interest. Consistent with the literature on sampled-data systems, we use a family of approximate discrete-time models

$$x(k+1) = F_{T,h}^a(x(k), u(k), w_T[k]), \tag{3}$$

where h is the parameter of the numerical integration and used to enhance the accuracy between the approximate discrete-time model (3) and the exact discrete-time model (2). Although one can simply take $h = T$ and use some classic approximation methods such as the Euler method, it is more appropriate to choose h different from T, as shown in [14] for instance.

The mismatch between the exact discrete-time model (2) and the approximate discrete-time model (3) is required to be small in the following sense.

Definition 1 (*[22,17]*). The family $F_{T,h}^a$ is said to be one-step consistent with F_T^e if for any real numbers $(\Delta_x, \Delta_u, \Delta_w)$ there exist a function $\rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that for each fixed $T \in (0, T^*)$ there exists $h^* \in (0, T]$ such that the following holds

$$|F_T^e(x, u, w_T) - F_{T,h}^a(x, u, w_T)| \leq T\rho(h)$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathcal{L}_\infty$ with $|x| \leq \Delta_x$, $|u| \leq \Delta_u$, $|w_T|_\infty \leq \Delta_w$ and all $h \in (0, h^*)$. □

Sufficient checkable conditions under which the one-step consistency is guaranteed are given in [13,14].

Assuming full access to the state x, we consider a family of controllers given by

$$u(k) = u_{T,h}(x_c(k)), \quad u_{T,h}(0) = 0, \tag{4}$$

where

$$x_c(k) = \begin{cases} x(k), \quad k = i\ell, i \in \mathbb{Z}_{\geq 0}, \\ F_{T,h}^a(x_c(k-1), u(k-1), 0), \text{ with } x_c(i\ell) = x(i\ell), \\ \quad \text{otherwise}. \end{cases} \tag{5}$$

The controller uses the sampled value of the state every T_s seconds. To compensate for the lack of information in the inter-samples the controller uses the estimated state values obtained from the disturbance-free plant model. Note that, if $\ell = 1$, the family of controllers (4) and (5) reduces to

$$u(k) = u_{T,h}(x(k)), \quad u_{T,h}(0) = 0. \tag{6}$$

The following definition introduces the notion of *uniform local boundedness* of $u_{T,h}$ [12,17] that is required in the proof of the main results (Theorems 1 and 2).

Definition 2 (*[17]*). The control law $u_{T,h}$ is said to be uniformly locally Lipschitz if for any $\Delta_x > 0$ there exist positive real numbers T^* and $\tilde{L}$ such that for each fixed $T \in (0, T^*)$ there exists $h^* \in (0, T]$ such that

$$|u_{T,h}(x_2) - u_{T,h}(x_1)| \leq \tilde{L}|x_2 - x_1|$$

for all $h \in (0, h^*)$ and all $x_1, x_2 \in \mathbb{R}^n$ with $\max\{|x_1|, |x_2|\} \leq \Delta_x$. □