Systems & Control Letters 117 (2018) 23–29

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Affine and bilinear systems on Lie groups

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ARTICLE INFO

Article history:
Received 13 February 2017
Received in revised form 5 March 2018
Accepted 1 May 2018

Keywords:
Affine systems
Bilinear systems
Controllability

ABSTRACT

In this paper we study affine and bilinear systems on Lie groups. We show that there is an intrinsic connection between the solutions of both systems. Such relation allows us to obtain some preliminary controllability results of affine systems on compact and solvable Lie groups. We also show that the controllability property of bilinear systems is very restricted and may only be achieved if the state space G is an Euclidean space.

1. Introduction

An affine system on a connected Lie group G is a family

$$\dot{x}(t) = F^0(x(t)) + \sum_{j=1}^{m} \omega_j(t) F^j(x(t)),$$

of ordinary differential equations, where $\omega := (\omega_1, \ldots, \omega_m) \in \mathcal{U}$ is a piecewise constant function and $F^0, F^1, \ldots, F^m$ are affine vector fields.

The class of affine systems are in fact quite large since it contains the classical linear and bilinear systems on the Euclidean space $\mathbb{R}^d$ and more generally the invariant, linear and bilinear systems on G (see [1–3] and [4]). Therefore, the dynamic involved here is really much more complicated than those of the mentioned systems.

In the present paper we exploit the intrinsic connection between affine and bilinear systems in order to obtain controllability results for affine systems. One example where one can see how strong is such connection is given for $G = \mathbb{R}^n$ by Jurdevic and Sallet in [5]. There the authors showed that an affine system is controllable as soon as it has no singularities and its associated bilinear system is controllable in $\mathbb{R}^n \setminus \{0\}$. However, any other class of Lie groups contains nontrivial proper subsets that are naturally invariant by automorphisms implying that controllability of any bilinear system on $G \setminus \{e\}$ can only be expected when G is isomorphic to $\mathbb{R}^n$ (see Theorem 3.4 ahead). Therefore, generalizations of the result of Jurdevic and Sallet for more general Lie groups are not possible.

The above forces us to look at affine systems in a more geometric way by using the above invariant subsets as done in [6] and [7] for linear systems. In order to do that we first prove that there is an intrinsic connection between the solutions of any affine system and its associated bilinear system. More accurate, the solutions of an affine system are given by left translation of the solutions of their associated bilinear system. Using such formula we are able to generalize some results from [7] allowing us to prove controllability results for affine systems on compact and solvable Lie groups under the assumption of local controllability around the identity.

This paper is structured as follows. In Section 2 we introduce the basic concepts about control systems and affine vector fields. In Section 3 we analyze bilinear systems on Lie groups. We give an explicit formula for the solutions of such systems and show that the controllability of bilinear systems is only to be expected in Euclidean spaces. Section 4 is devoted to the understanding of affine systems. We show that the solution of an affine system is given by left translation of the solution of its associated bilinear system. Such expression allows us to prove some results concerning the controllability of affine systems on compact and solvable Lie groups.

2. Preliminaries

In this section, we introduce basic concepts that will be needed through the paper.

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https://doi.org/10.1016/j.sysconle.2018.05.001
0167-6911/

2.1. Notations

By a smooth manifold we understand a finite-dimensional, connected, second-countable, Hausdorff manifold endowed with a $\mathcal{C}^\infty$-differentiable structure. If $f : M \to N$ is a differentiable map between smooth manifolds, we write $(df)_x : T_xM \to T_{f(x)}N$ for its derivative at $x \in M$, where T_xM is the tangent space at $x \in M$ and $T_{f(x)}N$ the tangent space at $f(x) \in N$. When we do not need to specify the point $x \in M$ we say only that f_* is the derivative of f.

A Lie group G will be a group endowed with the structure of a smooth manifold. If G is a Lie group, we write $\mathrm{Aut}(G)$ for the groups of automorphisms of G and $\mathfrak{X}(G)$ for the set of $\mathcal{C}^\infty$ vector fields on G. By e we denote the identity element of G and by i the inversion of G, that maps $g \in G$ into its inverse $g^{-1} \in G$. For any given $g \in G$ we denote by L_g, R_g and C_g the left translation, right translation and the conjugation by g, respectively. The image of the exponential map $\exp : \mathfrak{g} \to G$ is denoted by $\exp(X)$ or by e^X. The Lie algebra $\mathfrak{g}$ of G will always be identified with the set of right invariant vector fields on G.

2.2. Control systems

A control system on a smooth manifold M is given by the family

$$\dot{x}(t) = f^0(x(t)) + \sum_{j=1}^{m} \omega_j(t) f^j(x(t)), \quad \omega = (\omega_1, \dots, \omega_m) \in \mathcal{U}, \qquad (\Sigma)$$

of ordinary differential equations. Here $f^0, f^1, \dots, f^m$ are smooth vector fields on M. f^0 is called the *drift vector field* and $f^1, \dots, f^m$ the *control vector fields*. The set of *admissible control functions* $\mathcal{U}$ is given by the set of piecewise constant functions $\omega : \mathbb{R} \to \mathbb{R}^m$.

For each $\omega \in \mathcal{U}$, the corresponding differential equation Σ has a unique solution $\varphi(t, x, u)$ with initial value $x = \varphi(0, x, u)$. The systems considered in this paper all have globally defined solutions, which give rise to a map

$$\varphi : \mathbb{R} \times M \times \mathcal{U} \to M, \quad (t, x, \omega) \mapsto \varphi(t, x, \omega),$$

called the *transition map* of the system. We also use the notation $\varphi_{t,\omega}$ for the map $\varphi_{t,\omega} : M \to M$ given by $x \mapsto \varphi_{t,\omega}(x) := \varphi(t, x, \omega)$. Since $f^0, f^1, \dots, f^m$ are smooth, the map $\varphi_{t,\omega}$ is also smooth. The transition map φ is a cocycle over the shift flow

$$\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, (t, \omega) \mapsto \theta\omega = \omega(\cdot + t),$$

i.e., it satisfies $\varphi(t + s, x, \omega) = \varphi(s, \varphi(t, x, \omega), \theta_t\omega)$ for all $t, s \in \mathbb{R}$, $x \in M$ and $\omega \in \mathcal{U}$. Moreover, it holds that $\varphi_{t,\omega}^{-1} = \varphi_{-t,\theta_t\omega}$ and, for all $t_1, t_2 > 0$ and $\omega_1, \omega_2 \in \mathcal{U}$

$$\varphi(t_1, \varphi(t_2, x, \omega_2), \omega_1) = \varphi(t + s, x, \omega),$$

$$\text{where} \quad \omega(\tau) = \begin{cases} \omega_1(\tau) \text{ for } \tau \in [0, s] \\ \omega_2(\tau - s) \text{ for } \tau \in [s, t + s]. \end{cases}$$

For $x \in M$ and $\tau > 0$ we write

$$\mathcal{R}_{\leq\tau}(x) := \{\varphi(t, x, \omega);\ t \in [0, \tau] \text{ and } \omega \in \mathcal{U}\} \quad \text{ and}$$

$$\mathcal{R}(x) := \bigcup_{\tau>0} \mathcal{R}_{\leq\tau}(x)$$

for the *set of points reachable from* $x \in M$ *up to time* τ and the *reachable set from* x, respectively. Analogously, we define the *set of points controllable to* x *within time* τ and the *controllable set of* x respectively by

$$\mathcal{R}^*_{\leq\tau}(x) := \{y \in M; \exists t \in [0, \tau], \omega \in \mathcal{U} \text{ with } \varphi(t, y, \omega) = x\} \quad \text{ and}$$

$$\mathcal{R}^*(x) := \bigcup_{\tau>0} \mathcal{R}^*_{\leq\tau}(x).$$

The system Σ is said to be *locally controllable at* x if $x \in \operatorname{int}\mathcal{R}(x)$. In the analytic case, it follows from Theorem 3.1 of [8] that Σ is locally controllable at x if $x \in \operatorname{int}\mathcal{R}(x) \cap \operatorname{int}\mathcal{R}^*(x)$. In particular, that is the case for the systems on Lie groups under consideration in this paper. The system Σ is said to be *controllable in* $X \subset M$ if for all $x, y \in X$ there exist $\tau > 0$ and $\omega \in \mathcal{U}$ such that $y = \varphi(\tau, x, \omega)$. Equivalently, the system is controllable in $X \subset M$ if $X \subset \mathcal{R}(x) \cap \mathcal{R}^*(x)$ for some (and hence for all) $x \in X$.

Remark 2.1. It is worth to mention that the problem or characterizing local controllability was studied by many authors (see for instance Hermes [9,10] Sussmann [11–13] Bianchini and Stefani [14]). Necessary and sufficient conditions for local controllability are expressible in terms of $X \in \mathcal{L}$, where $\mathcal{L} = \mathcal{L}(f^0, f^1, \dots, f^m)$ denote the smallest Lie algebra of vector fields on M containing $f^0, f^1, \dots, f^m$. Indeed all the papers above give sufficient conditions for local reachability.

Remark 2.2. The choice of the set of admissible control functions being piecewise constant is not restrictive. In fact, most of the usual choices of admissible functions are such that the solutions of Σ can be approximated by using piecewise constant ones.

2.3. Affine and linear vector fields

In this section we define affine and linear vector fields and state their main properties. For the proof of the assertions in this section the reader can consult [1,15] and [16].

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Following [1], the *normalizer* of $\mathfrak{g}$ is the set

$$\eta := \{F \in \mathfrak{X}(G); \quad \text{for all } Y \in \mathfrak{g}, \quad [F, Y] \in \mathfrak{g}\}.$$

A vector field F on G is said to be *affine* if it belongs to η. If $F \in \eta$ and $F(e) = 0$ the vector field F is said to be *linear*. Any affine vector field F is uniquely decomposed as $F = \mathcal{X} + Y$ where $\mathcal{X}$ is linear and Y is right invariant. Moreover, any $F \in \eta$ is complete, any linear vector field $\mathcal{X}$ is an infinitesimal automorphism, that is, its flow in 1-parameter subgroup of $\mathrm{Aut}(G)$, and if $\{\alpha_t\}_{t\in\mathbb{R}}$ and $\{\psi_t\}_{t\in\mathbb{R}}$ stand, respectively, for the flow of F and $\mathcal{X}$, where $F = \mathcal{X} + Y$, we have that

$$\alpha_t(g) = L_{\alpha_t(e)}(\psi_t(g)), \quad \text{for all} \quad g \in G. \qquad (1)$$

The next technical lemma shows that expression (1) can be generalized for finite composition of flows of affine vector fields. Such result will be needed ahead.

Lemma 2.3. *Let* $\{F_i\}_{i\in\mathbb{N}}$ *be a family of affine vector fields with decomposition* $F_i = \mathcal{X}_i + Y_i$ *where* $\mathcal{X}_i$ *is linear and* Y_i *is right-invariant, for any* $i \in \mathbb{N}$. *Denote by* $\{\alpha^i_t\}_{t\in\mathbb{R}}$ *and* $\{\psi^i_t\}_{t\in\mathbb{R}}$ *the flows of* F_i *and* $\mathcal{X}_i$ *respectively. For any* $i_1, \dots, i_n \in \mathbb{N}$ *and any real numbers* $\tau_1, \dots, \tau_n$, *it holds that*

$$\alpha^{i_n}_{\tau_n} \circ \cdots \circ \alpha^{i_1}_{\tau_1} = L_{\alpha^{i_n}_{\tau_n}\left(\cdots\left(\alpha^{i_1}_{\tau_1}(e)\right)\cdots\right)} \circ \psi^{i_n}_{\tau_n} \circ \cdots \circ \psi^{i_1}_{\tau_1}. \qquad (2)$$

Proof. Our proof is by induction. For $n = 1$ such equation coincides with (1) and the result holds. Let us consider $i_1, \dots, i_{n+1} \in \mathbb{N}$, $\tau_1, \dots, \tau_{n+1}$ and by the hypothesis of induction assume that

$$\alpha^{i_n}_{\tau_n} \circ \cdots \circ \alpha^{i_1}_{\tau_1} = L_{\alpha^{i_n}_{\tau_n}\left(\cdots\left(\alpha^{i_1}_{\tau_1}(e)\right)\cdots\right)} \circ \psi^{i_n}_{\tau_n} \circ \cdots \circ \psi^{i_1}_{\tau_1}$$

holds. Hence,

$$\alpha^{i_{n+1}}_{\tau_{n+1}} \circ \alpha^{i_n}_{\tau_n} \circ \cdots \circ \alpha^{i_1}_{\tau_1} = \alpha^{i_{n+1}}_{\tau_{n+1}} \circ L_{\alpha^{i_n}_{\tau_n}\left(\cdots\left(\alpha^{i_1}_{\tau_1}(e)\right)\cdots\right)} \circ \psi^{i_n}_{\tau_n} \circ \cdots \circ \psi^{i_1}_{\tau_1}$$

$$= L_{\alpha^{\tau_{n+1}}_{i_{n+1}}(e)} \circ \psi^{i_{n+1}}_{\tau_{n+1}} \circ L_{\alpha^{i_n}_{\tau_n}\left(\cdots\left(\alpha^{i_1}_{\tau_1}(e)\right)\cdots\right)} \circ \psi^{i_n}_{\tau_n} \circ \cdots \circ \psi^{i_1}_{\tau_1}.$$