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Design of state-feedback control for polynomial systems with quadratic performance criterion and control input constraints

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ABSTRACT

This paper presents a novel convex optimization approach to design state-feedback control for polynomial systems. Design criteria are comprised of a quadratic cost function and bounded magnitudes of control inputs. Specifically, we formulate a control synthesis of closed-loop systems operated in a given bounded domain characterized by a semi-algebraic set. We consider an extended class of rational Lyapunov functions and derive an upper bound of the cost function, together with a state-feedback control law. By exploiting bounds on the control input magnitudes, the controller design condition can be cast as a parameter-dependent linear matrix inequality (PDLMI), which is convex optimization and can be efficiently solved by sum-of-squares (SOS) technique. In addition, we derive a sufficient condition to compute a lower bound of the cost function. When choosing polynomial structure of the solution candidate, the lower bound can also be written as PDLMI. Numerical examples are provided to illustrate the effectiveness of the proposed design.

1. Introduction

Controller synthesis of nonlinear systems has received considerable attention for several decades. When focusing on the class of *polynomial systems*, i.e., nonlinear systems characterized by polynomial vector fields, various Lyapunov synthesis problems were formulated as convex optimization problems called *parameter-dependent linear matrix inequalities (PDLMIs)* [1,2], by applying the notion of state-dependent linear-like representations [3–6] and considering some certain classes of Lyapunov functions [5–10]. Although PDLMIs are difficult to solve, one can approximate them as standard LMIs via sum-of-squares (SOS) technique [2,8,11].

In the previous works of [3,4,12,13], state-feedback synthesis with local asymptotic stability was established using quadratic or polynomial Lyapunov functions and can be cast as PDLMIs. For local stable synthesis, rational Lyapunov functions [5–7] are more flexible than polynomial Lyapunov functions, and hence result in better closed-loop performance than those obtained from polynomial Lyapunov functions. Construction of rational Lyapunov functions, however, is generally difficult since the problem is originally nonconvex [7,14] and cannot be directly formulated as PDLMIs. The nonconvex issue were partially addressed by several approaches [4–6]. Such approaches rely on a very limited class of Lyapunov functions [4], or imposing some additional constraints to the original nonlinear systems [5,6]. It turns out that the extra constraints are difficult to verify in practice.

This paper aims to design state-feedback control for polynomial systems with a guaranteed upper bound on a quadratic cost function and subjected to bounded control inputs. It is noted that the control input constraints make the synthesis problem more practical, but harder in real control applications. Existing studies consider stabilizing control design dealing with input bounds (see [12] for example). We hardly see in the literature control design with a quadratic cost and control input constraints. In this paper, we develop a convex condition for designing a state-feedback controller which minimizes an upper bound on the cost function. The design condition is formulated as an PDLMI by considering an extended class of *rational Lyapunov functions*, and by exploiting the input magnitude bounds. To the best of our knowledge, the technique of exploiting input bounds to address non-convex design conditions has not been proposed in the literature. Main contributions of this paper are summarized as follows.
(i) We propose an effective approach to design a certain type of state-feedback control law. Indeed, the proposed design condition provides more flexibility in the construction of rational Lyapunov functions than those in [4–6]. In addition to our preliminary work in [15], we show that the proposed design condition using rational Lyapunov functions is less conservative than the existing design

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condition in [3], when the input constraints are imposed to the design problem. Numerical results illustrate the benefit of the proposed design condition and are compared to that of [3] when the input constraints are tight. Moreover, we provide an alternative design condition which requires less number of constraints than that of the main design.

(ii) We propose a functional inequality to obtain a lower bound on the cost function. Seeking a solution of this inequality is computationally tractable. In particular, the computation can be formulated as PDLMI by limiting the class of solution candidate to polynomials. Once the upper and lower bounds are computed, we can estimate how close the suboptimal performance is to the optimal performance. In other words, we can measure conservatism of the computed bounds. It is shown by numerical examples that the gap between the two bounds becomes smaller when we increase the degree of rational functions and polynomials. Moreover, when this gap is small enough, we can say that the suboptimal performance is close to the optimal performance.

This paper is organized as follows: Problem formulation is stated in Section 2. Section 3 explains state-feedback control design and upper bound condition. A lower bound of the quadratic cost is derived in Section 4. Section 5 presents numerical examples. Conclusions are given in Section 6.

2. Problem statement

Consider a class of nonlinear dynamical systems

$$\dot{x} = f(x) + B(x)u, \quad f(0) = 0$$

which can be represented in the following *state-dependent linear-like form*:

$$\dot{x} = A(x)Z(x) + B(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input. The vector $Z(x)$, whose dimension is assumed to be N, contains polynomials of x and satisfies the assumption $Z(x) = 0 \Leftrightarrow x = 0$. The matrices $A(x)$ and $B(x)$ are polynomial matrices of suitable dimensions.

For the above system, we aim to design a state-feedback controller $u = u(x)$ which belongs to the given input set $\mathcal{U}$. Additionally, we assume that the closed-loop system is operated on a compact domain $\mathcal{X} \subset \mathbb{R}^n$ containing the origin. Throughout this paper, we assume that $\mathcal{X} = \{x \mid Z(x)^{\mathrm{T}} B_{\mathcal{X}} Z(x) \le 1\}$ with a given $B_{\mathcal{X}} \succ O$. Since $Z(x)$ can contain polynomials of higher degrees in x, the region $\mathcal{X}$ is not necessarily a standard ellipsoid but it represents a more complicated operating region. As a performance measure under the input u, we consider the quadratic cost function

$$J(x_0, u) = \int_0^\infty (Z(x)^{\mathrm{T}} Q Z(x) + u^{\mathrm{T}} R u)dt, \tag{2}$$

with given matrices $Q \succ O$, $R \succ O$, and the initial condition $x(0) = x_0$.

Let the optimal quadratic cost be $J^*(x_0)$, i.e.,

$$J^*(x_0) = \min_{u \in \mathcal{U}} J(x_0, u), \quad \forall x_0 \in \mathcal{X},$$

and let the input set $\mathcal{U}$ is characterized by the hypercube

$$\mathcal{U} = \{u \in \mathbb{R}^m \mid |u_j| \le \mu_j, \quad j = 1, 2, \ldots, m\},$$

which represents bounds on input magnitudes. Moreover, we estimate the region of attraction for the closed-loop system using a level set of Lyapunov function [16], which is defined by

$$\Omega = \{x \in \mathbb{R}^n \mid V(x) \le \rho\},$$

where $V(x)$ is a Lyapunov function of the closed-loop system and ρ is a positive constant. Since both $V(x)$ and ρ are variables in the design problem, we can assume without loss of generality that $\rho = 1$.

To guarantee that the controller $u = u(x)$ will remain in the set $\mathcal{U}$ for any $x \in \Omega$, the following condition is required:

$$\Omega \subset X_{\mathcal{U}}, \tag{3}$$

where $X_{\mathcal{U}} \triangleq \{x \mid |u_j(x)| \le \mu_j, \quad j = 1, 2, \ldots, m\}$ is the set of all states such that the corresponding control inputs satisfy the magnitude constraints characterized by $\mathcal{U}$.

Computing the optimal cost $J^*(x_0)$, as well as the associate optimal controller, leads to solving a Hamilton–Jacobi–Bellman partial differential equation [17], which is computationally difficult. Therefore, we may instead find a controller that provides a finite upper bound on the quadratic cost. This task can be done via Lyapunov inequality as described in the following proposition (see, for example, [17] for a proof).

Proposition 1. *If there exist a continuously differentiable function $V : \mathcal{X} \to \mathbb{R}$, and a function $\hat{u} : \mathcal{X} \to \mathbb{R}^m$ satisfying $V(x) > 0$ ($\forall x \in \mathcal{X} \setminus \{0\}$), $V(0) = 0$, and*

$$\begin{aligned} &\frac{\partial V(x)}{\partial x}^{\mathrm{T}}(A(x)Z(x) + B(x)\hat{u}) + \\ &Z(x)^{\mathrm{T}} Q Z(x) + \hat{u}^{\mathrm{T}} R \hat{u} \le 0, \qquad \forall x \in \mathcal{X}, \end{aligned} \tag{4}$$

then the closed-loop system is asymptotically stable with respect to the zero equilibrium. Moreover, if the set-inclusion constraints (3) and $\Omega \subset \mathcal{X}$ hold, then for any initial condition x_0 inside the level set Ω, it yields that $J^*(x_0) \le J(x_0, \hat{u}) \le V(x_0)$.

Note that searching for Lyapunov function $V(x)$ and controller $u(x)$ satisfying (4) is still a difficult task. Nevertheless, choosing a suitable characterization of $V(x)$ and an input set $\mathcal{U}$ can lead to a convex condition for the controller design. Details will be explained in the next section.

Remark 1. Based on the assumption $f(0) = 0$, it is always possible to find a polynomial matrix $A(x)$ such that $f(x) = A(x)x$. Moreover, the representation $f(x) = A(x)x$ can be rewritten as

$$f(x) = \begin{bmatrix} A(x) & O \end{bmatrix} \begin{bmatrix} x \\ \tilde{Z}(x) \end{bmatrix}$$

where $\tilde{Z}(x)$ contains monomials or polynomials of higher degrees in x. This implies that the state-dependent linear-like representation (1) always exists, where choice of $Z(x)$ is not unique.

When $Z(x)$ is fixed, it is not difficult to show that choice of $A(x)$ is also not unique. Precisely speaking, if the representation (1) exists for a polynomial matrix $A_0(x)$, then it also exists for $A(x) = A_0(x) + N(x)$, where $N(x)$ is a non-zero polynomial matrix satisfying $N(x)Z(x) = O$.

3. State-feedback control

In order to find $V(x)$ and $u(x)$ satisfying constraints in Proposition 1, we choose a candidate Lyapunov function of the form

$$V(x) = Z(x)^{\mathrm{T}} P^{-1}(x) Z(x), \tag{5}$$

where $P(x)$ is a symmetric polynomial matrix of dimension $N \times N$ in x and $P(x) \succ O$, $\forall x \in \mathcal{X}$. Moreover, the controllers are parametrized by $u = K(x)Z(x)$ with a proper matrix $K(x)$. Note here that $V(x)$ and $K(x)$ are extended to rational functions of x. For Lyapunov function of the form (5), the corresponding level set is

$$\Omega(P) = \{x \in \mathbb{R}^n \mid Z(x)^{\mathrm{T}} P^{-1}(x) Z(x) \le 1\}.$$

We put the argument $P(x)$ in the notation of Ω to indicate that the shape and the size of the level set depend on $P(x)$.