Systems & Control Letters 116 (2018) 27–31

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Can a finite number of discrete delays approximate stochastic delay?

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ARTICLE INFO

Article history:
Received 29 April 2017
Received in revised form 20 January 2018
Accepted 15 April 2018

Keywords:
Stochastic delay
Discrete delay
Stability
Linear systems

ABSTRACT

In stability analysis and control design for a system with stochastic delay, it is a question whether one can approximate the stochastic system, for instance in the sense of average, with a deterministic system that has a finite number of discrete delay terms with the same delays that appear in the stochastic system and the weight coefficients of these delayed terms are taken from the probability distribution function of the stochastic delay. In this note, we consider a linear system with stochastic delay and discuss conditions under which this approximation is valid and conditions where it is not. In particular, we assume that the delay has piece-wise constant realizations with constant dwelling time at each value and show that the above mentioned approximation loses its grounds when the delay dwelling time gets larger than the minimum delay in the system.

Published by Elsevier B.V.

1. Introduction

Consider the linear system

$$\dot{x}(t) = a\,x(t) + b\,x(t-\tau(t)), \tag{1}$$

where the delay $\tau(t)$ stochastically changes in a finite set $\Omega = \{\tau_1, \tau_2, \ldots, \tau_K\}$ while it dwells at each value a fixed amount of time $t_{\rm d}$. In particular, the changes in the delay occur at times $nt_{\rm d}$, $n = 0, 1, 2, \ldots$, while new delay values are chosen according to the probability distribution $\mathbb{P}\big(\tau(t) = \tau_k\big) = w_k$, $k = 1, \ldots, K$.

Consider also the deterministic system

$$\dot{x}(t) = a\,x(t) + b\sum_{k=1}^{K} w_k\,x(t-\tau_k) \tag{2}$$

which contains, in the right hand side, a finite number of discrete (point) delay terms where the delays are the same as in the set Ω and the weight coefficients of the discrete delay terms are the same as the probability distribution w_k, $k = 1, \ldots, K$.

In analyzing the stability of the stochastic system (1) in engineering applications, it is a question whether one could consider the deterministic system (2) as an approximation for the mean of the stochastic system (1). For instance in connected vehicle systems, stochastic delays arise due to the random packet loss in wireless communication between vehicles [1]. Similarly in networked control systems, the communication delays in wireless communication channels may change stochastically in time [2,3]. In this paper, we show that the approximation of the mean of the stochastic system (1) by the deterministic system (2) can be completely misleading.

In particular, we consider the linear system (1) where $a, b \in \mathbb{R}$ and assume that the delay $\tau(t)$ can only take two delay values τ_1 and τ_2 where $0 < \tau_1 < \tau_2$; a sample realization of the delay is shown in Fig. 1a. The delay dwells in one value for a duration of $t_{\rm d}$ (dwelling time) and then switches to a new value based on the probability distribution $\mathbb{P}(\tau = \tau_1) = w_1$, $\mathbb{P}(\tau = \tau_2) = w_2$, where $w_1 + w_2 = 1$. Using this simplistic behavior for the delay, we aim to show that the dwelling time $t_{\rm d}$ can have a substantial effect on the stability of the mean of the stochastic system (1) that cannot be captured by the corresponding deterministic system (2). The use of scalar versions of systems (1) and (2) and the assumption that the delay can assume only two values are for the sake of the brevity of the notation and clarity in conveying the message of the paper. The results of the paper hold for the general vector case (*i.e.* $x \in \mathbb{R}^q$ and $a, b \in \mathbb{R}^{q\times q}$ where q is the dimension of vector x) with arbitrary $K \in \mathbb{N}$ delays in the set Ω. Our approach is to use a suitable time-discretization of the two systems (1) and (2) and compare the stability of these systems through comparing the spectra of the matrices emerging from the time-discretization of the two systems.

2. Discretization and approximation of spectra

In this section, we obtain a time discretization of both systems (1) and (2) which we will later use to compare the stability of

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https://doi.org/10.1016/j.sysconle.2018.04.007
0167-6911/Published by Elsevier B.V.

these systems. To this aim, we first present the method of the discretization of a deterministic system with a single fixed delay using the semi-discretization technique developed in [4]. The semi-discretization technique is a well-known time discretization technique for delay differential equations and widely-used in engineering applications [4–6].

2.1. *Discretization of a system with a single fixed delay*

Consider the following deterministic system with a single fixed delay

$$\dot{x}(t) = a\,x(t) + b\,x(t-\tau). \tag{3}$$

By substituting the trial solution $x(t) = \kappa \mathrm{e}^{\lambda t}$ in Eq. (3), we may obtain the corresponding characteristic equation

$$\lambda - a - b\,\mathrm{e}^{-\lambda\tau} = 0 \tag{4}$$

where λ is a characteristic root. System (3) is stable if and only if all characteristic roots are located in the left-half complex plane.

Now consider the mesh $t_i = i\Delta t$, $i = 0, 1, 2, \ldots$. Let $m = \lfloor \tau/\Delta t \rfloor$. Now in the time interval $[i\Delta t, (i+1)\Delta t]$, we use the approximation $x(t-\tau) \approx x(i\Delta t - m\Delta t)$ in (3) and solve the resulting ordinary differential equation $\dot{x}(t) = a\,x(t) + b\,x\big((i-m)\Delta t\big)$ to obtain

$$x\big((i+1)\Delta t\big) = \alpha\, x(i\Delta t) + \beta\, x\big((i-m)\Delta t\big), \tag{5}$$

where $\alpha = \mathrm{e}^{a\Delta t}$ and $\beta = \frac{b}{a}(\mathrm{e}^{a\Delta t} - 1)$, $\big(\beta = b\Delta t$ if $a = 0\big)$. Now forming an augmented state vector $X(i) = \big[x(i\Delta t),\ x\big((i-1)\Delta t\big),\ \ldots,\ x\big((i-m)\Delta t\big)\big]^{\mathrm{T}}$, (T denotes the transpose) that contains a history of the state values in the last m time steps, we arrive at

$$X(i+1) = T(\Delta t)X(i), \tag{6}$$

where

$$T(\Delta t) = \begin{bmatrix} \alpha & & & & \beta \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{bmatrix}_{(m+1)\times(m+1)}. \tag{7}$$

Note that all sub-diagonal elements are 1 and the 0 elements are not shown. The matrix $T(\Delta t)$ can be called the evolution matrix of system (3) since it is a finite-dimensional approximation of the infinite-dimensional solution operator of the linear system (3) [4].

Assume μ is an eigenvalue of $T(\Delta t)$. Then, as $\Delta t \to 0$, $\frac{1}{\Delta t}\ln\mu$ approaches a characteristic root given by (4); *i.e.* $\frac{1}{\Delta t}\ln\mu \to \lambda$. Therefore, one can obtain stable and unstable regions of system (3) in the parameter space by investigating the leading eigenvalues (largest eigenvalues in magnitude) of $T(\Delta t)$ that are calculated for a sufficiently small Δt value. Note that the larger the magnitude of μ the bigger the real part of $\frac{1}{\Delta t}\ln\mu$. See [4] for more details about the convergence properties of the semi-discretization method and [7] for more information on the discretization of the delay differential equations and approximating their spectra using other numerical techniques. In the next section, we investigate the difference in stability properties of systems (1) and (2) exploiting the semi-discretization of the two systems.

2.2. *Stability of systems (1) and (2) using their finite-dimensional approximations*

We first apply the discretization method described in the previous section to the stochastic system (1). We choose Δt such that

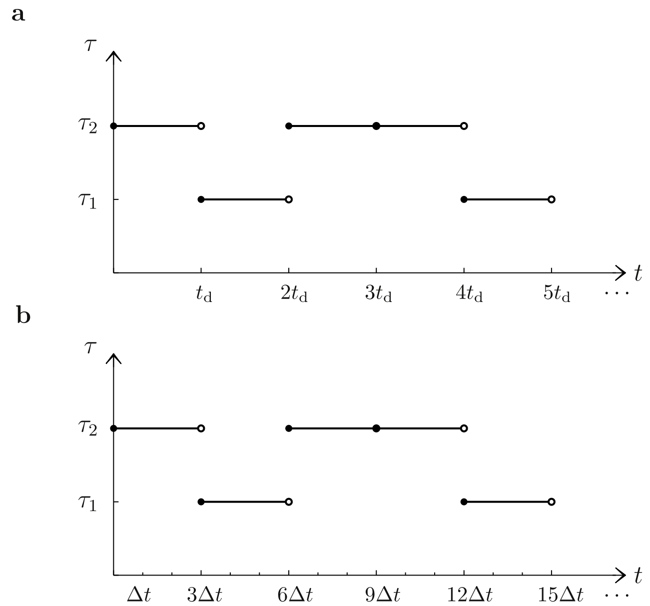

Fig. 1. (a) A sample path of the delay with two values τ_1 and τ_2 and dwelling time t_{d}. (b) The dwelling time t_{d} is discretized to ℓ time steps such that $t_{\mathrm{d}} = \ell\Delta t$ ($\ell = 3$ in this case).

$t_{\mathrm{d}} = \ell\Delta t$ where ℓ is an integer. We also assume $m_1 = \lfloor \tau_1/\Delta t \rfloor$ and $m_2 = \lfloor \tau_2/\Delta t \rfloor$ where $m_1 < m_2$. Using the augmented state vector $X(i) = \big[x(i\Delta t),\ x\big((i-1)\Delta t\big), \ldots, x\big((i-m_2)\Delta t\big)\big]^{\mathrm{T}}$, that contains a history of the state values in the last m_2 time steps (corresponding to the maximum delay), and similar to (6) and (7), the evolution matrix of the system $\dot{x}(t) = a\,x(t) + b\,x(t-\tau_1)$ is obtained as

$$T_1(\Delta t) = \begin{array}{c} \text{column } m_1+1 \\ \downarrow \\ \begin{bmatrix} \alpha & & \beta & & \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{bmatrix}_{(m_2+1)\times(m_2+1)} \end{array}, \tag{8}$$

and the evolution matrix of the system $\dot{x}(t) = a\,x(t) + b\,x(t-\tau_2)$ is obtained as

$$T_2(\Delta t) = \begin{array}{c} \text{column } m_2+1 \\ \downarrow \\ \begin{bmatrix} \alpha & & & & \beta \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{bmatrix}_{(m_2+1)\times(m_2+1)} \end{array}, \tag{9}$$

where α and β are the same as in (5). Now recall that the delay changes every ℓ time steps; see Fig. 1b. Therefore, letting $\tilde{X}(n) = X(n\,\ell)$, the discretization of the stochastic system (1) is given by the stochastic map

$$\tilde{X}(n+1) = A(n)\tilde{X}(n), \tag{10}$$

$n = 0, 1, 2, \ldots$, where $A(n) = \big(T_k(\Delta t)\big)^{\ell}$ if $\tau(t) = \tau_k$ in the time interval $[n\ell\Delta t, (n+1)\ell\Delta t)$, and therefore $\mathbb{P}\Big(A(n) = \big(T_k(\Delta t)\big)^{\ell}\Big) = w_k$, $k = 1, 2$.

Now we take the expectation of both sides of (10). Since the probability distribution of the delay is fixed and is independent of