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Unique solvability of a singular stochastic control model for population management

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ABSTRACT

We present and mathematically analyze a singular stochastic control model for cost-effective and ecologically-sound indirect population control strategy for fish-eating birds. Finding the optimal strategy of a threshold type reduces to solving a variational inequality. We prove the unique existence of its viscosity solution that is neither convex nor concave, which turns out to be a classical solution. Comparative statics on the optimal threshold is performed as well to demonstrate practical implications of the model.

1. Introduction

Stochastic control theory provides simple and efficient mathematical models for management of systems in a feed-back manner [1–3]. The concept of singular stochastic control has especially been discussed for management of resources to which a threshold-type control strategy is appropriate [4–6]. Finding an optimal strategy in the context of stochastic singular control reduces to solving a variational inequality (VI) having viscosity solutions [7].

A central issue in modern stochastic control theory is existence and solvability of dynamic programming equations, nonlinear and degenerate elliptic differential equations that govern the value functions and associated optimal controls [7]. This is not a trivial mathematical issue as the mathematical analyses carried out in previous research on seemingly simple mathematical models suggest [1–6]. This issue can be resolved by utilizing the concept of viscosity solutions, which possess nice mathematical properties such as certain stability and regularity [7].

This letter presents mathematical results on a conceptual singular stochastic control problem for indirect population control of the fish-eating bird such as *Phalacrocorax Carbo*, which is an urgent issue to be solved in inland fisheries [8,9]. The goal of the problem is to find a management strategy for the bird population so that its predation to fishery resources is suppressed while they do not become extinct. A mathematical difficulty in dealing with the model is that the value function is neither convex nor concave, which is different from in the conventional problems (Chapter 4.5 of [7]). This motivated us to employ a new constructive argument to prove unique solvability of the VI. We show that the VI is uniquely solvable in the viscosity sense. Practical implications of the present model are briefly presented as well. This paper thus contributes to both theory and application of a singular stochastic control model.

The rest of this paper is organized as follows. Section 2 formulates our stochastic control problem and derives the VI, which is the main equation of this paper. Section 3 presents a classical solution to the VI, but its uniqueness is not a trivial matter at this stage. Section 4 then presents a series of mathematical analysis results to prove the unique solvability of the VI in the viscosity sense, and shows that the classical solution is indeed the unique viscosity solution to the VI. A key here is a constructive argument to restrict possible profiles of viscosity solutions. Section 5 concludes this paper and presents future perspectives of our research. A supplementary file focusing on a practical problem related to the VI is also attached.

2. Mathematical model

We consider a cost-effective and ecologically-sound indirect population control strategy for a population of a fish-eating bird. The indirect population control here means not to kill the individuals, but to effectively decrease growth rate of the population [10]. The time is denoted as $t \in [0, \infty)$. The bird population dynamics is

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described by an Itô's stochastic differential equation (SDE) with a non-negative, unbounded, right-continuous, and adapted control variable η_t for $t \geq 0$ (Chapter 4.5.1 of [7]). The SDE to be controlled is based on that of a geometric Brownian motion:

$$\mathrm{d}X_t = X_t\left(\mu \mathrm{d}t + \sigma \mathrm{d}B_t - \mathrm{d}\eta_t\right), \quad t \geq 0, \; X_{-0} = x \geq 0 \tag{1}$$

where X_t is the population size of the bird at t, $\mu > 0$ is the deterministic growth rate of the population, $\sigma > 0$ is the magnitude of stochastic fluctuation of the population dynamics, B_t is the standard 1-D Brownian motion. In the SDE (1), η_t represents decrease of the growth rate of the population through the indirect population control. The SDE (1) has a control variable that is singular in the sense that it is neither continuous nor bounded.

The objective function to be maximized by the decision-maker, the manager of the fish, with an appropriate η_t is set as

$$J(x, \eta) = -\int_0^{\infty} e^{-\delta s}\left(rX_s^m - RX_s^M\right)\mathrm{d}s - \int_0^{\infty} e^{-\delta s}\mathrm{d}\eta_s \tag{2}$$

where $\delta > 0$ is the discount rate, r and R are the parameters such that $r > R > 0$, and m and M are the parameters such that $0 < M < 1 < m < 2$. The discount rate δ represents the attitude of the decision-maker; larger (smaller) δ means that the decision-maker performs the population control from a long-term (short-term) viewpoint. This J represents the net profit of the decision-maker. The term $-rX_s^m$ quantifies the loss of the fish by the bird per unit time and RX_s^M represents the ecosystem services per unit time that the bird possibly provides [11,12]. The second term in the right-hand side of (2) represents the cost of performing the indirect population control.

The value function $\Phi(\cdot)$ is the maximized objective function in the sense of expectation:

$$\Phi(x) = \sup_{\eta} \mathrm{E}\left[J(x, \eta)\right] = \mathrm{E}\left[J\left(x, \eta^*\right)\right], \tag{3}$$

where $\mathrm{E}[\cdot]$ is the expectation conditioned on $X_{-0} = x \geq 0$ and η^* is an optimal control. According to the dynamic programming principle (Chapter 4.3 of [7]), $\Phi(\cdot)$ formally solves the VI

$$\min\left[-L\Phi + rx^m - Rx^M, x\frac{\mathrm{d}\Phi}{\mathrm{d}x} + 1\right] = 0, \;\; x > 0 \tag{4}$$

subject to the boundary condition at the origin $\Phi(0) = 0$ where $L = \mu x \frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\sigma^2 x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \delta$ is the generator. The boundary condition means that there is no profit and loss when there is no population. Hereafter, the assumptions

$$2\mu > \sigma^2 \quad \text{and} \\ \delta > \mu m + \frac{\sigma^2}{2}m(m-1)\left(> \mu M + \frac{\sigma^2}{2}M(M-1)\right) \tag{5}$$

is employed. In other words, the population does not become extinct when there is no intervention and the decision-maker wants to control the bird population from a long-term viewpoint. The following lemma is a consequence of the property of the process (1) and the definition of J, which can be proved with a straightforward calculation. The lemma means that Φ is locally bounded and has at most a polynomial growth rate.

Lemma 2.1.

$$Ax^m + Bx^M \leq \Phi(x) \leq \frac{R(m-M)}{\delta m}\left(\frac{RM}{rm}\right)^{\frac{M}{m-M}}, \quad x \geq 0 \tag{6}$$

where

$$A = \frac{-r}{\delta - \mu m - \frac{\sigma^2}{2}m(m-1)} < 0, \\ B = \frac{R}{\delta - \mu M - \frac{\sigma^2}{2}M(M-1)} > 0. \tag{7}$$

3. Exact solution of the VI

We explore a classical solution $\Phi \in C^2(0, \infty) \cap C[0, \infty)$ to the VI (4) of the form

$$\Phi(x) = \begin{cases} a_1 x^{k_1} + a_2 x^{k_2} + A' x^m + B' x^M & (0 < x \leq \bar{x}) \\ b - \log x & (x > \bar{x}) \end{cases} \tag{8}$$

where $a_1, a_2, A', B', b, \bar{x}, k_1 = \frac{1}{2}\left(1 - \frac{2\mu}{\sigma^2} + \sqrt{\left(\frac{2\mu}{\sigma^2} - 1\right)^2 + \frac{8\delta}{\sigma^2}}\right) > 0$, $k_2 = \frac{1}{2}\left(1 - \frac{2\mu}{\sigma^2} - \sqrt{\left(\frac{2\mu}{\sigma^2} - 1\right)^2 + \frac{8\delta}{\sigma^2}}\right) < 0$ are constants. The solution (8) means that the indirect population control should be activated when the bird population X_t reaches the threshold value $\bar{x}$. The polynomial $a_1 x^{k_1} + a_2 x^{k_2}$ is the general solution of $-L\Phi = 0$ and $A'x^m + B'x^M$ is the particular solution of $-L\Phi + rx^m - Rx^M = 0$. A direct calculation shows $A' = A$, $B' = B$, and $k_1 > m$: the last one follows from (5). Lemma 2.1 shows that $\lim_{x \to +0}\Phi(x)$ is bounded and thus $a_2 = 0$ since $k_2 < 0$. Hereafter, the descriptions $k_1 = k$ and $a_1 = a$ are employed for the sake of brevity. The solution (8) was constructed based on the conjecture that there exists some threshold value of x above which the population should be controlled [1,4,6,7].

The assumed regularity of Φ at $x = \bar{x}$ leads to

$$\begin{cases} a\bar{x}^k + A\bar{x}^m + B\bar{x}^M = b - \log \bar{x}, \\ ka\bar{x}^{k-1} + mA\bar{x}^{m-1} + MB\bar{x}^{M-1} = -\bar{x}^{-1}, \\ k(k-1)a\bar{x}^{k-2} + m(m-1)A\bar{x}^{m-2} \\ \quad + M(M-1)B\bar{x}^{M-2} = \bar{x}^{-2}, \end{cases} \tag{9}$$

with the three unknowns a, b, and $\bar{x}$. Combining the second and third equations of (9) yields the equation of $\bar{x}$:

$$\bar{x}^m = \frac{-\left[MB\bar{x}^M(k-M) + k\right]}{mA(k-m)}. \tag{10}$$

The classical intermediate value theorem to (10) shows that there is unique $0 < \bar{x} < \infty$ that solves (10). Substituting this $\bar{x}$ into the second equation in (9) uniquely determines a. Then, substituting this $\bar{x}$ and a into the first equation in (9) uniquely determines b. Consequently, the next theorem follows.

Theorem 3.1. *There exists unique viscosity solution $\Phi \in C^2(0, \infty) \cap C[0, \infty)$ of the form (8) that solves the VI (4).*

Remark 3.2. Φ in (8) is neither convex nor concave in $(0, \infty)$ since it is concave for small $x > 0$ and convex for $x > \bar{x}$.

Our supplementary file analyses parameter dependence of $\bar{x}$ for interested readers.

4. Unique solvability of the VI

We show that the exact classical solution (8) is a unique viscosity solution to the VI (4) that satisfies the boundary condition $\Phi(0) = 0$ in the pointwise sense. The proofs below are inspired from those for an optimal investment problem having a concave value function (Chapter 4.5 of [7]), but include different mathematical techniques. In what follows, we carry out mathematical analysis of the VI (4) from the standpoint that the value function is its viscosity solution, and show that it is actually a unique classical solution to the VI (4) and is analytically expressed as (8) based on a constructive argument. We focus on continuity viscosity solutions, which are natural candidates of solutions to VIs in singular stochastic control problems [4,6,7]. For definitions of viscosity sub- and super-differentials utilized in what follows, see [13].