Systems & Control Letters 115 (2018) 1–8

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Stability of highly nonlinear neutral stochastic differential delay equations

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ARTICLE INFO

Article history:
Received 7 April 2017
Received in revised form 12 February 2018
Accepted 26 February 2018

Keywords:
Neutral stochastic differential delay equation
Nonlinear growth condition
Asymptotic stability
Delay dependent
Markovian switching

ABSTRACT

Stability criteria for neutral stochastic differential delay equations (NSDDEs) have been studied intensively for the past several decades. Most of these criteria can only be applied to NSDDEs where their coefficients are either linear or nonlinear but bounded by linear functions. This paper is concerned with the stability of hybrid NSDDEs without the linear growth condition, to which we will refer as highly nonlinear ones. The stability criteria established in this paper will be dependent on delays.

1. Introduction

Many stochastic dynamical systems do not only depend on present and past states but also involve derivatives with delays. Neutral stochastic differential delay equations (NSDDEs) are often used to model such systems. NSDDEs with Markovian switching (also known as hybrid NSDDEs) form an important class of hybrid dynamical systems. They have been successfully applied in practice, such as in traffic control, switching power converters, neural networks, and so on (see, e.g., [1–5]). The research on the stability of NSDDEs with Markovian switching has received considerable attention for the past several decades (see, e.g., [6–9]). The stability criteria are in general classified into two categories: delay-dependent and delay-independent stability criteria. The delay-dependent stability criteria take into account the size of delays and hence are generally less conservative than the delay-independent ones which work for any size of delays.

A common feature of the existing delay-dependent stability criteria is that most of them can only be applied to delay systems where their coefficients are either linear or nonlinear but bounded by linear functions (see, e.g., [10–12]). However, the linear growth condition is usually violated in many practical applications. Recently, there are some progress on stability for highly nonlinear stochastic delay systems. For example, the stability and boundedness of nonlinear hybrid SDDEs were studied in [13], the robust stability and boundedness of SDDEs without the linear growth condition were studied in [14], the stability of neutral stochastic differential equations with unbounded delay and Markovian switching was studied in [15]. But those results are all delay independent. [16] is the first to establish delay-dependent criteria for highly nonlinear hybrid SDDEs. However, to the best of our knowledge, there is so far no delay-dependent stability criteria for highly nonlinear hybrid NSDDEs. Motivated by [16], the key aim of this paper is to establish the delay-dependent stability criteria for hybrid NSDDEs with the polynomial growth condition instead of the linear growth condition. To explain our aim more clearly, let us consider the scalar highly nonlinear hybrid NSDDE

$$d[x(t) - D(x(t-\tau))] = f(x(t), x(t-\tau), r(t), t)dt + g(x(t), x(t-\tau), r(t), t)dB(t), \qquad (1.1)$$

where $x(t) \in \mathbb{R}$ is the state, τ stands for time delay, $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $\mathbb{S} = \{1, 2\}$ with its generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \qquad (1.2)$$

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https://doi.org/10.1016/j.sysconle.2018.02.013
0167-6911/

and the coefficients are defined by

$$\begin{aligned} &f(x,y,1,t) = -y - 4x^3, \quad f(x,y,2,t) = -y - 5x^3, \\ &g(x,y,1,t) = g(x,y,2,t) = 0.5y^2, \\ &D(x(t-\tau)) = 0.1x(t-\tau). \end{aligned} \tag{1.3}$$

This nonlinear hybrid NSDDE can be regarded as that it operates in two modes and it obeys

$$\begin{aligned} d[x(t) - 0.1x(t-\tau)] =& [-x(t-\tau) - 4x^3(t)]dt \\ &+ 0.5x^2(t-\tau)dB(t), \end{aligned}$$

$$\begin{aligned} d[x(t) - 0.1x(t-\tau)] =& [-x(t-\tau) - 5x^3(t)]dt \\ &+ 0.5x^2(t-\tau)dB(t) \end{aligned}$$

in mode 1 and 2, respectively. The system will switch from one mode to the other according to the probability law of the Markov chain. If $\tau = 0.01$, the computer simulation shows that the hybrid NSDDE is asymptotically stable. If the time-delay is large, say $\tau = 2$, the computer simulation shows that the hybrid NSDDE is unstable. In other words, whether the hybrid NSDDE is stable or not depends on how small or large the time-delay is. On the other hand, both drift and diffusion coefficients of the hybrid NSDDE are highly nonlinear. Unfortunately, there is so far no delay-dependent criterion which can be applied to this NSDDE to derive a sufficient bound on the time-delay τ for the NSDDE to be stable. Our aim here is to establish delay-dependent criteria for such highly nonlinear hybrid NSDDEs.

2. Notation and assumption

Throughout this paper, unless otherwise specified, we use the following notation. If A is a vector or matrix, its transpose is denoted by A^T. If $x \in \mathbb{R}^n$, then $|x|$ is its Euclidean norm. For a matrix A, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^TA)}$. For $\tau > 0$, denote by $C([-\tau, 0]; \mathbb{R}^n)$ the family of continuous functions φ from $[-\tau, 0] \to \mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau \le u \le 0} |\varphi(u)|$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \ldots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$, Here $\gamma_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ while $\gamma_{ii} = -\sum_{j\ne i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. Let

$$\begin{aligned} &f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n, \\ &g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}, \quad D : \mathbb{R}^n \to \mathbb{R}^n \end{aligned}$$

be Borel measurable functions. Consider an n-dimensional hybrid NSDDE

$$\begin{aligned} d[x(t) - D(x(t-\tau))] =& f(x(t), x(t-\tau), r(t), t)dt \\ &+ g(x(t), x(t-\tau), r(t), t)dB(t) \end{aligned} \tag{2.1}$$

on $t \ge 0$ with initial data

$$\{x(t) : -\tau \le t \le 0\} = \eta \in C([-\tau, 0]; \mathbb{R}^n), \; r(0) = i_0 \in \mathbb{S}. \tag{2.2}$$

The well-known conditions imposed for the existence and uniqueness of the global solution are the local Lipschitz condition and the linear growth condition (see, e.g., [4,9,17]). In this paper, we need the local Lipschitz condition. However, we impose the polynomial growth condition instead of the linear growth condition. Let us state these conditions as an assumption for the use of this paper.

Assumption 2.1. Assume that for any $h > 0$, there exists a positive constant K_h such that

$$\begin{aligned} |f(x,y,i,t) - f(\bar{x},\bar{y},i,t)| \vee |g(x,y,i,t) - g(\bar{x},\bar{y},i,t)| \\ \le K_h(|x-\bar{x}| + |y-\bar{y}|) \end{aligned} \tag{2.3}$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le h$ and all $(i, t) \in \mathbb{S} \times \mathbb{R}_+$. Assume also that there exist three constants $K > 0$, $q_1 \ge 1$ and $q_2 \ge 1$ such that

$$\begin{aligned} &|f(x,y,i,t)| \le K(1 + |x|^{q_1} + |y|^{q_1}), \\ &|g(x,y,i,t)| \le K(1 + |x|^{q_2} + |y|^{q_2}) \end{aligned} \tag{2.4}$$

for all $(x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$. Assume moreover that there is a constant $\kappa \in (0, \frac{\sqrt{2}}{2})$ such that

$$|D(u) - D(v)| \le \kappa|u - v| \tag{2.5}$$

for all $u, v \in \mathbb{R}$, and $D(0) = 0$.

Of course, if $q_1 = q_2 = 1$, then condition (2.4) is the familiar linear growth condition. However, we emphasize once again that we are here interested in highly nonlinear NSDDEs which have either $q_1 > 1$ or $q_2 > 1$. We will refer to condition (2.4) as the polynomial growth condition. Of course, without the linear growth condition, the solution of the NSDDE (2.1) may explode to infinity at a finite time. To avoid such a possible explosion, we need to impose an additional condition in terms of Lyapunov functions. For this purpose, we need more notation. Let $C^{2,1}(\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$ denote the family of non-negative functions $U(x, i, t)$ defined on $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$ which are continuously twice differentiable in x and once in t. We now state another assumption.

Assumption 2.2. Assume that there exists a pair of functions $\bar{U} \in C^{2,1}(\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$ and $G \in C(\mathbb{R}^n \times [-\tau, \infty); \mathbb{R}_+)$, as well as positive numbers c_1, c_2, c_3 and $q \ge 2(q_1 \vee q_2)$, such that

$$c_3 < c_2, \quad |x|^q \le \bar{U}(x,i,t) \le G(x,t),$$

$\forall (x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, and

$$\begin{aligned} &\mathbb{L}\bar{U}(x - D(y), y, i, t) : \\ &\quad = \bar{U}_t(x - D(y), i, t) + \bar{U}_x(x - D(y), i, t) f(x, y, i, t) \\ &\quad + \frac{1}{2}\text{trace}[g^T(x,y,i,t)\bar{U}_{xx}(x - D(y), i, t) g(x, y, i, t)] \\ &\quad + \sum_{j=1}^{N} \gamma_{ij} \bar{U}(x - D(y), j, t) \\ &\quad \le c_1 - c_2 G(x, t) + c_3 G(y, t - \tau), \end{aligned}$$

$\forall (x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

We now cite a result from [15] as a lemma for the use of this paper.

Lemma 2.3. *Under Assumptions 2.1 and 2.2, the NSDDE (2.1) with the initial data (2.2) has the unique global solution $x(t)$ on $t \ge -\tau$ and the solution has the property that* $\sup_{-\tau \le t < \infty} \mathbb{E}|x(t)|^q < \infty$.

3. Delay-dependent asymptotic stability

In this section, we will use the method of Lyapunov functionals to investigate the delay-dependent asymptotic stability. We define two segments $\bar{x}_t := \{x(t+s) : -2\tau \le s \le 0\}$ and $\bar{r}_t := \{r(t+s) : -2\tau \le s \le 0\}$ for $t \ge 0$. For $\bar{x}_t$ and $\bar{r}_t$ to be well defined for $0 \le t < 2\tau$, we set $x(s) = \eta(-\tau)$ for $s \in [-2\tau, -\tau)$ and $r(s) = r_0$