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Positive realizations of nonlinear input–output discrete-time systems

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ABSTRACT

The problem of realization of a positive nonlinear input–output discrete-time system as a positive state-space system is investigated. The input–output system is given by a difference equation of a higher order. A necessary and sufficient condition for existence of positive and regularly positively observable realization is provided. The state space variables of the realization are found as independent generators of some positive universe associated to the input–output system. Only the single-input single-output case is studied and all functions describing the systems and their properties are assumed to be continuous.

1. Introduction

A realization problem in system theory consists in finding a state-space representation of an input–output system where only inputs and outputs appear. There are two major descriptions of input–output systems: response maps (or input–output maps) and equations of higher order. In this paper we concentrate on the latter description. More specifically, we search for positive nonlinear realizations of positive input–output systems given by higher order difference equations. This study may be seen as a counterpart of [1], where we investigated positive realizations of positive response maps. The two descriptions of input–output systems differ in many ways. So do realizations problems. In particular, a realization of a response map involves an initial point, which does not appear in a realization of an equation of higher order. In the nonlinear case, passing from one description to the other is usually not possible, so translating a solution of one realization problem to a solution of the other seems to be out of reach.

Positive systems naturally appear in biology, chemistry or economics. In such systems the variables take only nonnegative values. Examples can be found e.g. in [2] and [3], where also a theory of linear positive systems was developed. Positive realizations in the linear case are now well understood (see e.g. [4–9]). Both continuous- and discrete-time cases have been addressed. The external descriptions included input–output maps, input–output equations of higher order and transfer matrices. Realization theory of linear positive systems on time scales was developed in [9]. It unifies continuous- and discrete-time theories. Positive nonlinear systems on time scales appeared in [1]. The language of time scales allowed for unified treatment of the positive realization problem for response maps. The same procedure can be used in continuous- and discrete-time cases. The case of higher order nonlinear equations seems to be harder to be solved in such generality, so we focus in this paper on discrete-time systems. To our knowledge, problems of nonlinear positive realizations, in continuous- or discrete-time, have not been studied before.

The main result states necessary and sufficient conditions for a positive nonlinear input–output difference equation of higher order to have a positive and regularly positively observable state-space realization. Regular positive observability means that the state can be recovered as a positive function of n initial outputs and $n-1$ initial inputs, where n is the dimension of the state space. It is shown that for regularly positively observable state-space systems one can eliminate the state variables and obtain a positive input–output system whose positive external behavior coincides with positive external behavior of the state-space system. So this type of observability seems to be essential for the positive realization problem. However one may try to weaken this property as it is only sufficient for passing from the state-space description to the input–output one.

In Section 3 we define the crucial concept of positive universe. Then we construct a decreasing sequence of positive universes related to the input–output system. The structure of the last universe allows to decide whether a positive realization exists. The generators of this universe are then the state variables of the realization. As in [1] the language of function universes plays important role, but unlike in [1] the explicit concept of positive universe is used. Both positive universe and global function universe exploited in [1] are simplified versions of the concept of universe introduced by J. Johnson [10,11]. A global or positive universe is a generalization of algebra. In algebra we can substitute elements into polynomials of several variables; in a global/positive universe we can substitute

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elements into functions from a larger class, like the class of analytic or continuous functions.

The idea of characterizing positive realizability with the aid of certain positive universes is borrowed from [12–14], where spaces of differential one-forms play a similar role. Unfortunately, differentials of functions (i.e. exact one-forms) do not preserve positivity, so it seems that the approach based on spaces of one-forms is not suitable for studying positive systems. Thus we rather exploit functions and not their differentials, losing all tools that were developed in the differential approach. Actually no differential appears in the main part of the paper as we assume only continuity of the functions that describe the systems and their properties.

As in [1] the criteria of realizability are not very constructive. But even in the linear case, finding positive realizations is much harder and less constructive than finding general realizations, not necessarily positive, [8,3,9]. Since a necessary condition for existence of positive realization is existence of some general, not necessarily positive, realization, we present two results on existence of general realizations. The first is a simple modification of our criterion for positive realizability. In particular, positive universes are replaced with global function universes. This may be seen as a new characterization of general realizability of nonlinear input–output systems. The second characterization is a citation from [14]. It uses spaces of one-forms and needs the assumption of analyticity of the systems. Thus before we check existence of a positive realization, we can first check if any realization exists. To avoid technical complications we study only the single-input single-output case. The same approach can be used in developing realization theory of positive systems with many inputs and many outputs.

Let us fix notation that will be used in this paper. By $\mathbb{R}$ we shall denote the set of all real numbers, by $\mathbb{Z}$ the set of integers, and by $\mathbb{N}$ the set of natural numbers (without 0). The set of nonnegative real numbers will be denoted by $\mathbb{R}_+$ and the set of nonnegative integers by $\mathbb{Z}_+$. Similarly, $\mathbb{R}^k_+$ will mean the set of all column vectors in $\mathbb{R}^k$ with nonnegative components.

For $k \in \mathbb{N}$ let $\mathfrak{P}_k$ denote the set of all real nonnegative continuous functions defined on $\mathbb{R}^k_+$. Let $\mathfrak{P}$ be the disjoint union of all $\mathfrak{P}_k$ for $k \in \mathbb{N}$.

For a function $z : \mathbb{Z} \to X$, where X is an arbitrary set, we define $z^+ : \mathbb{Z} \to X$ by $z^+(k) := z(k+1)$. The same applies to z defined on $\mathbb{Z}_+$.

2. Positive systems

Let us consider a discrete-time system

$$\Sigma : \quad x(k+1) = f(x(k), u(k)), \;\; y(k) = h(x(k)) \tag{1}$$

where $k \in \mathbb{Z}_+$, the state $x(k) \in \mathbb{R}^n$, the output $y(k) \in \mathbb{R}$ and the input $u(k) \in \mathbb{R}$. The maps f and h are assumed to be continuous. We shall often write Eqs. (1) in a shorter form

$$x^+ = f(x, u), \;\; y = h(x).$$

Definition 2.1. The *behavior* of system Σ is the set of all solutions $(y(k), u(k), x(k))_{k\in\mathbb{Z}_+}$ satisfying (1). The set of nonnegative solutions is called *positive behavior*. The *external behavior* of system Σ is the set of all sequences $(y(k), u(k))_{k\in\mathbb{Z}_+}$, such that there is a sequence $(x(k))_{k\in\mathbb{Z}_+}$ such that $(y(k), u(k), x(k))_{k\in\mathbb{Z}_+}$ belongs to the behavior of system Σ. The *positive external behavior* of system Σ, denoted by $EB^+(\Sigma)$, is the set of all nonnegative sequences $(y(k), u(k))_{k\in\mathbb{Z}_+}$, such that there is a sequence $(x(k))_{k\in\mathbb{Z}_+}$ such that $(y(k), u(k), x(k))_{k\in\mathbb{Z}_+}$ belongs to the positive behavior of system Σ.

Definition 2.2. We say that the system Σ is *positive* if for any initial condition $x_0 \in \mathbb{R}^n_+$, any input u with $u(k) \in \mathbb{R}^m_+$, $k \in \mathbb{Z}_+$, the solution x of $x(k+1) = f(x(k), u(k))$ satisfies $x(k) \in \mathbb{R}^n_+$ and the output $y(k) \in \mathbb{R}^p_+$ for $k \in \mathbb{Z}_+$.

For $\omega \in \mathbb{R}^m$ let $f_\omega : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $f_\omega(x) := f(x, \omega)$. Then we have a simple well known characterization of positivity.

Proposition 2.3. *System Σ is positive if and only if for every $\omega \in \mathbb{R}^m_+$ the set $\mathbb{R}^n_+$ is invariant with respect to f_ω and $h(\mathbb{R}^n_+) \subseteq \mathbb{R}^p_+$.*

Let us consider now an input–output discrete-time system given by the equation

$$\Xi : \; y(k+n) = \Phi(y(k), \ldots, y(k+n-1), u(k), \ldots, u(k+n-1)), \tag{2}$$

where $n \in \mathbb{N}$ is fixed, $\Phi : E \to \mathbb{R}$, $E \subseteq \mathbb{R}^n \times \mathbb{R}^n$ has a nonempty interior, and $k \in \mathbb{Z}_+$. Φ is assumed to be continuous.

Definition 2.4. System Ξ is called *positive*, if $E \subseteq \mathbb{R}^n_+ \times \mathbb{R}^n_+$ and Φ is nonnegative on E.

Definition 2.5. The *external behavior* of system Ξ is the set of all solutions $(y(k), u(k))_{k\in\mathbb{Z}_+}$ of (2). The set of nonnegative solutions of (2) is called *positive external behavior* of system Ξ and denoted by $EB^+(\Xi)$.

The sequences belonging to external behaviors of Ξ or Σ will be often called *external trajectories* or just *trajectories*.

Let us go back to the state-space description.

Definition 2.6. A positive system Σ is *regularly positively observable* if there is a continuous function $g : D \to \mathbb{R}^n_+$, where $D \subseteq \mathbb{R}^n_+ \times \mathbb{R}^{n-1}_+$, such that for every nonnegative input u, every initial state $x(0) \in \mathbb{R}^n_+$ can be recovered from the input u and the output y corresponding to u and $x(0)$ as $x(0) = g(y(0), \ldots, y(n-1), u(0), \ldots, u(n-2))$. This in particular means that the point $(y(0), \ldots, y(n-1), u(0), \ldots, u(n-2)) \in D$. We shall assume that D consists of all such points. For $n = 1$ we identify $\mathbb{R}^n_+ \times \mathbb{R}^{n-1}_+$ with $\mathbb{R}^n_+$. Then g depends only on $y(0)$.

Proposition 2.7. *Assume that a positive system Σ is regularly positively observable. Then its positive external behavior coincides with the positive external behavior of some positive input–output system of the form*

$$\Xi : \; y(k+n) = \Phi_\Sigma(y(k), \ldots, y(k+n-1), u(k), \ldots, u(k+n-1)),$$

where Φ_Σ is defined on some $E \subseteq \mathbb{R}^n_+ \times \mathbb{R}^n_+$ and is continuous.

Proof. Let $(y(k), u(k))_{k\in\mathbb{Z}_+}$ belong to $EB^+(\Sigma)$. It is uniquely determined by $(u(k))_{k\in\mathbb{Z}_+}$, and the initial state $x(0) \in \mathbb{R}^n_+$. Then $y(0) = h(x(0)) =: H_0(x(0))$, $y(1) = h(f(x(0), u(0))) =: H_1(x(0), u(0))$,..., $y(n) = H_n(x(0), u(0), \ldots, u(n-1))$, where the function H_n is obtained through suitable compositions of h and f. Replacing $x(0)$ with $x(k)$ and $u(i)$ with $u(i+k)$ for $k \geq 0$ and repeating the calculations we obtain the relation

$$y(k+n) = H_n(x(k), u(k), \ldots, u(k+n-1)), \tag{3}$$

where $(x(k))_{k\in\mathbb{Z}_+}$ is the state trajectory determined by $x(0)$ and $(u(k))_{k\in\mathbb{Z}_+}$. Observe that, from positivity of Σ, the function H_n takes nonnegative values if $x(k), u(k), \ldots, u(k+n-1)$ are nonnegative. The same can be done with the relation $x(0) = g(y(0), \ldots, y(n-1), u(0), \ldots, u(n-2))$ describing regular positive observability. If