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Observability of Boolean networks via set controllability approach☆

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ABSTRACT

Controllability and observability of Boolean control networks (BCNs) are two fundamental properties. But verification of the latter is much harder than the former. This paper considers the observability of BCNs via controllability. First, set controllability is proposed, and a necessary and sufficient condition is obtained. Then a technique is developed to convert observability into an equivalent set controllability problem. Using the result for set controllability, a necessary and sufficient condition is also obtained for the observability of BCNs.

1. Introduction

Boolean network was firstly proposed by Kauffman to describe gene regularity networks [1]. Since then it has attracted much attention from biologists, physicists, and system scientists [2,3].

Recently, a new matrix product, called semi-tensor product (STP) of matrices was introduced. STP has then been successfully applied to modeling and controlling Boolean networks [4–6]. Inspired by the STP, theory of BCNs as well as the control of general logical systems has been developed rapidly. A set of systematic results has been obtained. For instance, controllability and observability of Boolean networks have been discussed in [7,8]; the disturbance decoupling has been considered in [9,10]; the optimal control has been investigated in [11,12]; the stability and stabilization have been studied in [13,14], just to mention a few.

Among them controllability and observability of BCNs are of particular importance. Particularly, controllability via free control sequence is fundamental, and it has been solved elegantly by [15]. Unlike controllability, observability has also been discussed for long time and various kinds of observability have been proposed and investigated [7,15,16]. A comparison for various kinds of observability has been presented in [17]. Moreover, [17] has also pointed out that one of them, which will be specified later, is the most sensitive observability. Here “most sensitive one” means that as an observability criterion it is most sensitive. That is, this observability requires least conditions. Hence any other observability in literature, which requires more, implies this observability. In addition, [17] has also provided necessary and sufficient conditions for various kinds of observability via finite automata approach. Motivated by the idea of [17], [18] proposed a numerical method to verify the (most sensitive) observability.

Since for BCNs controllability is much easier understandable and verifiable than observability, this paper proposes a method to verify observability via controllability. In this paper set controllability of BCNs is proposed first. The idea comes from [8], where some states are forbidden and it is a special case of our set controllability. Hence the result about set controllability can be considered as a generalization of the corresponding result in [8]. Then a properly designed extended system of the original BCN is built. It is proved that observability of the original BCN is equivalent to set controllability of the extended system. Then observability of the original BCN is converted into a set controllability problem and then is solved completely. In fact, the result is equivalent to the necessary and sufficient condition proposed in [18]. But the new result is concise and easily verifiable.

The rest of this paper is organized as follows: Section 2 describes the set controllability of BCNs. The set controllability matrix is constructed. Using it an easily verifiable necessary and sufficient condition is obtained. As an application, output controllability problem is also solved. Section 3 constructs an extended system and carefully designs the initial and destination sets. Then the observability of a BCN becomes the set controllability of its extended system. An example is presented to describe the design procedure. Section 4 is a concluding remark.

Before ending this section, a list of notations is presented:

(1) $\mathbb{R}^n$: n dimensional Euclidean space.
(2) $\mathcal{M}_{m\times n}$: the set of $m \times n$ real matrices.

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(3) Col(M): the set of columns of M. Col$_i(M)$: the ith column of M.
(4) $\mathcal{D} := \{0, 1\}$.
(5) δ_n^i: the ith column of the identity matrix I_n.
(6) $\Delta_n := \{\delta_n^i | i = 1, \ldots, n\}$.
(7) $\mathbf{1}_\ell := (\underbrace{1, 1, \ldots, 1}_{\ell})^{\mathrm{T}}$, $\mathbf{1}_{p\times q} := (\underbrace{\mathbf{1}_p, \mathbf{1}_p, \ldots, \mathbf{1}_p}_{q})$.
(8) A matrix $L \in \mathcal{M}_{m\times n}$ is called a logical matrix if Col(L) $\subset \Delta_m$. Denote by $\mathcal{L}_{m\times n}$ the set of $m \times n$ logical matrices.
(9) If $L \in \mathcal{L}_{n\times r}$, by definition it can be expressed as $L = [\delta_n^{i_1}, \delta_n^{i_2}, \ldots, \delta_n^{i_r}]$. For the sake of compactness, it is briefly denoted as $L = \delta_n[i_1, i_2, \ldots, i_r]$.
(10) Denote by $\mathcal{B}_{m\times n}$ the set of $m \times n$ Boolean matrices.
(11) $M +_{\mathcal{B}} N = A$ is the Boolean addition of $M, N \in \mathcal{B}_{m\times n}$, where $a_{i,j} = m_{i,j} \vee n_{i,j}$.
(12) $M \times_{\mathcal{B}} N = A$ is the Boolean product of $M \in \mathcal{B}_{r\times s}, N \in \mathcal{B}_{s\times t}$, where $a_{i,j} = \sum_{\mathcal{B}k=1}^{s} m_{i,k} \wedge n_{k,j}$.
(13) $A \in \mathcal{B}_{n\times n}$, then $A^{(k)} := \underbrace{A \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} A}_{k}$.
(14) A matrix $C > 0$ means all the entries are positive, that is, $c_{i,j} > 0, \forall i, j$.

2. Set controllability

A Markov-type Boolean control network with n nodes is described as [4]

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \ldots, x_n(t); u_1(t), \ldots, u_m(t)) \\ x_2(t+1) = f_2(x_1(t), \ldots, x_n(t); u_1(t), \ldots, u_m(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \ldots, x_n(t); u_1(t), \ldots, u_m(t)), \end{cases} \\ y_j(t) = h_j(x_1(t), \ldots, x_n(t)), \quad j = 1, \ldots, p, \tag{1}$$

where $x_i \in \mathcal{D}, i = 1, \ldots, n$ are state variables; $u_i \in \mathcal{D}, i = 1, \ldots, m$ are controls; $y_j \in \mathcal{D}, j = 1, \ldots, p$ are outputs; $f_i : \mathcal{D}^{m+n} \to \mathcal{D}$, $i = 1, \ldots, n$, and $h_j : \mathcal{D}^n \to \mathcal{D}, j = 1, \ldots, p$ are Boolean functions.

Definition 2.1 ([4]). The system (1) is

1. controllable from x_0 to x_d, if there are a $T > 0$ and a sequence of controls $u(0), \ldots, u(T-1)$, such that driven by these controls the trajectory can go from $x(0) = x_0$ to $x(T) = x_d$;
2. controllable at x_0, if it is controllable from x_0 to destination $x_d = x, \forall x$;
3. controllable, if it is controllable at any x.

Under the vector form expression:

$1 \sim \delta_2^1, 0 \sim \delta_2^2$,

we have $x_i,\ u_i,\ y_j \in \Delta_2$. Then (1) can be converted into its algebraic form as [4]

$$\begin{cases} x(t+1) = Lu(t)x(t) \\ y(t) = Hx(t), \end{cases} \tag{2}$$

where $x(t) = \ltimes_{i=1}^n x_i(t)$, $y(t) = \ltimes_{i=1}^p y_i(t)$, $u(t) = \ltimes_{j=1}^m u_j(t)$, and $L \in \mathcal{L}_{2^n\times 2^{n+m}}$, $H \in \mathcal{L}_{2^p\times 2^n}$.

Define

$$M := \sum_{\mathcal{B}\ j=1}^{2^m} L\delta_{2^m}^j, \tag{3}$$

and set

$$\mathcal{C} := \sum_{\mathcal{B}\ i=1}^{2^n} M^{(i)}, \tag{4}$$

which is called the controllability matrix. Then we have the following result:

Proposition 2.2 ([15]). *Consider the controllability of system (1) (by free control sequence). Assume its controllability matrix is $\mathcal{C} = (c_{i,j})$, then we have the following results:*

1. *State $x_d = \delta_{2^n}^i$ is controllable from $x_0 = \delta_{2^n}^j \neq \delta_{2^n}^i$, if and only if, $c_{i,j} = 1$.*
2. *System (1) is controllable at $x = \delta_{2^n}^j$, if and only if,* Col$_j(\mathcal{C}) = \mathbf{1}_{2^n}$.
3. *System (1) is controllable, if and only if, $\mathcal{C} = \mathbf{1}_{2^n\times 2^n}$.*

Denote by $N = \{1, 2, \ldots, 2^n\}$ the set of states. Assume $s \in 2^N$, the index vector of s, denoted by $V(s) \in \mathbb{R}^{2^n}$, is defined as

$$(V(s))_i = \begin{cases} 1, & i \in s \\ 0, & i \notin s. \end{cases}$$

Define the family of initial sets P^0 and the family of destination sets P^d respectively as follows:

$$\begin{aligned} P^0 &:= \{s_1^0, s_2^0, \ldots, s_\alpha^0\} \subset 2^N, \\ P^d &:= \{s_1^d, s_2^d, \ldots, s_\beta^d\} \subset 2^N \end{aligned} \tag{5}$$

where $\emptyset \notin P^0$ and $\emptyset \notin P^d$.

Using initial sets and destination sets, the set controllability is defined as follows.

Definition 2.3. Consider system (1) with a family of initial sets P^0 and a family of destination sets P^d. The system (1) is

1. set controllable from $s_j^0 \in P^0$ to $s_i^d \in P^d$, if there exist $x_0 \in s_j^0$ and $x_d \in s_i^d$, such that x_d is controllable from x_0;
2. set controllable at s_j^0, if for any $s_i^d \in P^d$, the system is controllable from s_j^0 to s_i^d;
3. set controllable, if it is set controllable at any $s_j^0 \in P^0$.

Using the family of initial sets and the family of destination sets defined in (5), we can define the initial index matrix J_0 and the destination index matrix J_d respectively as

$$\begin{aligned} J_0 &:= \begin{bmatrix} V(s_1^0) & V(s_2^0) & \cdots & V(s_\alpha^0) \end{bmatrix} \in \mathcal{B}_{2^n\times\alpha}; \\ J_d &:= \begin{bmatrix} V(s_1^d) & V(s_2^d) & \cdots & V(s_\beta^d) \end{bmatrix} \in \mathcal{B}_{2^n\times\beta}. \end{aligned} \tag{6}$$

Using (6), we define a matrix, called set controllability matrix, as

$$\mathcal{C}_S := J_d^T \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{B}} J_0 \in \mathcal{B}_{\beta\times\alpha}. \tag{7}$$

Note that hereafter all the matrix products, used to calculate the set controllability matrices, are assumed to be Boolean product ($\times_{\mathcal{B}}$). Hence the symbol $\times_{\mathcal{B}}$ is omitted.

According to the definition of set controllability, the following result is easily verifiable.

Proposition 2.4. *Consider system (1) with the set of initial sets P^0 and the set of destination sets P^d as defined in (5). Moreover, the corresponding set controllability matrix $\mathcal{C}_S = (c_{ij})$ is defined in (7). Then*

1. *system (1) is set controllable from s_j^0 to s_i^d, if and only if, $c_{i,j} = 1$;*
2. *system (1) is set controllable at s_j^0, if and only if* Col$_j(\mathcal{C}_S) = \mathbf{1}_\beta$;
3. *system (1) is set controllable, if and only if, $\mathcal{C}_S = \mathbf{1}_{\beta\times\alpha}$.*

As an application, we consider output controllability [19]. Similarly, this approach may also be used for some other related problems, say, output regulation [20].

Definition 2.5. Consider system (1). It is said to be output controllable, if for any $x(0) = x_0$ and any y_d, there exist a $T > 0$ and a sequence of controls $u(0), u(1), \ldots, u(T-1)$ such that $y(T) = y_d$.