Systems & Control Letters 114 (2018) 27–30

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

A sufficient stochastic maximum principle for a kind of recursive optimal control problem with obstacle constraint

Jianhui Huang [a], Haiyang Wang [b], Zhen Wu [c,*]

[a] Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong
[b] School of Mathematical Sciences, Shandong Normal University, Jinan 250014, China
[c] School of Mathematics, Shandong University, Jinan 250100, China

ARTICLE INFO

Article history:
Received 3 August 2017
Received in revised form 1 January 2018
Accepted 6 February 2018

Keywords:
Maximum principle
Reflected backward stochastic differential equations
Recursive optimal control problems
Mixed control problems
Clarke's generalized gradient

ABSTRACT

In this paper, we study a kind of recursive optimal control problem whose utility functional is described by the solution of a reflected backward stochastic differential equation (BSDE). We obtain a sufficient stochastic maximum principle of optimal controls. Moreover, a mixed optimal control problem is considered to illustrate the application of our theoretical result and the optimal control and stopping strategy are given.

1. Introduction

Nonlinear backward stochastic differential equations (BSDEs) have been introduced by Pardoux and Peng [1]. They gave the general Feynman–Kac formula for BSDEs coupled with SDEs in [2]. Independently, Duffie and Epstein [3] presented a stochastic differential recursive utility, which is a generalization of standard additive utility with an instantaneous utility depending on not only the instantaneous consumption rate but also the future utility. As found by El Karoui et al. [4], the utility process can be regarded as a solution of BSDE. Reflected backward stochastic differential equations are a special kind of BSDEs, where a continuous increasing process is introduced to push the solution upward in a kind of minimal way, in order to keep it above a given stochastic process, called the "obstacle". El. Karoui et al. [5] got the existence and uniqueness of solutions to such reflected BSDEs in two different ways. This kind of reflected BSDEs has a broad range of applications in mathematical finance, optimal controls and so on. For example, El Karoui et al. [6] showed that the prices of American options correspond to the solutions of reflected BSDEs. Hamadène and Lepeltier [7] used reflected BSDEs to solve a mixed optimal control problem.

Wu and Yu [8] studied one kind of recursive optimal control problem with obstacle constraint. To be precise, the utility functional is described by the solution of a reflected BSDE with one lower obstacle. They proved that the dynamic programming principle still holds for this problem under the condition that the control domain is compact. Moreover, if the problem is formulated within a Markovian framework, the value function is shown to be the unique viscosity solution of an obstacle problem for nonlinear parabolic PDE which is called the Hamilton–Jacobi–Bellman (HJB) equation. Motivated by its significance of both theory and application mentioned above, we attempt to study a sufficient maximum principle of optimality for this kind of control problem with obstacle constraint, which is also efficient for the mixed optimization problems. We give such an example with numerical computation to illustrate the theoretical result.

The rest of this paper is organized as follows. In next section, we formulate a recursive optimal control problem with obstacle constraint. In Section 3, we give a sufficient maximum principle of optimality under convexity and continuity conditions. A mixed optimal control problem is studied in Section 4 to illustrate our theoretical result obtained above, where the optimal control and stopping strategy can be determined. We also provide a numerical example to demonstrate the optimal result. The last section is devoted to conclude the novelty and distinctive feature of this paper.

2. Formulation of the optimal control problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a natural filtration $\mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\}$, where $\{W_t\}_{t\geq 0}$ is a

* Corresponding author.
E-mail addresses: majhuang@polyu.edu.hk (J. Huang), wanghy@sdnu.edu.cn (H. Wang), wuzhen@sdu.edu.cn (Z. Wu).

https://doi.org/10.1016/j.sysconle.2018.02.005
0167-6911/

d-dimensional standard Brownian motion defined on the space. $T > 0$ is a fixed time horizon, and $\mathcal{F} = \mathcal{F}_T$. We denote by $\langle\cdot\rangle$ (respectively, $|\cdot|$) the scalar product (respectively, norm) for a given Euclidean space. Moreover, we define the admissible control set $\mathcal{U}$ by

$$\mathcal{U} := \Big\{\{v_t\}_{0\le t\le T} \text{ is a } U\text{-valued } \mathcal{F}_t\text{-adapted process} \quad \text{s.t. } \mathbb{E}\Big[\int_0^T |v_t|^2 dt\Big] < +\infty\Big\},$$

where U is a nonempty subset of $\mathbb{R}^l$. An element of $\mathcal{U}$ is called an admissible control.

For a given admissible control, consider the following stochastic control system:

$$\begin{cases} dx_t^v = b(t, x_t^v, v_t)dt + \sigma(t, x_t^v, v_t)dW_t, \quad 0 \le t \le T, \\ x_0^v = \alpha, \end{cases} \tag{1}$$

where $\alpha \in \mathbb{R}^d$ is a given constant, and the mappings $b(t, x, v) : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$, and $\sigma(t, x, v) : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times d}$ satisfy the following assumption:
(H2.1) b, σ are continuously differentiable in x for all $(t, v) \in [0, T] \times U$ and there exists a constant $L > 0$ such that

$$|b(t, x, v)| \le L(1 + |x|), \quad |\sigma(t, x, v)| \le L(1 + |x|),$$
$$|b(t, x, v) - b(t, x', v)| + |\sigma(t, x, v) - \sigma(t, x', v)| \le L|x - x'|.$$

Thus, by Theorem 5.2.1 of [9] (see also Theorem 6.3 in Chapter 1 of [10]), (1) admits a unique solution $\{x_t^v, 0 \le t \le T\}$ for any $v \in \mathcal{U}$.

Moreover, let us consider the following controlled reflected BSDE with the "obstacle" $\{h(t, x_t^v), 0 \le t \le T\}$:

$$y_t^v = g(x_T^v) + \int_t^T f(s, x_s^v, y_s^v, v_s)ds + k_T^v - k_t^v - \int_t^T z_s^v dW_s, \quad 0 \le t \le T, \tag{2}$$

where the mappings $f(t, x, y, v) : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times U \to \mathbb{R}$, $h(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}$, and $g(x) : \mathbb{R}^d \to \mathbb{R}$ satisfy the following assumption:
(H2.2) f, g and h are continuously differentiable in (x, y) for all $(t, v) \in [0, T] \times U$, $h(T, x) \le g(x)$ for all $x \in \mathbb{R}^d$, and there exists a constant $L > 0$ such that

$$|f(t, x, y, v)| \le L(1 + |x| + |y|), \quad |h(t, x)| \le L(1 + |x|),$$
$$|g(x)| \le L(1 + |x|),$$
$$|f(t, x, y, v) - f(t, x', y', v)| + |h(t, x) - h(t, x')| + |g(x) - g(x')| \le L(|x - x'| + |y - y'|).$$

Then by Theorem 5.2 in [5], for any $v \in \mathcal{U}$, (2) admits a unique solution $\{(y_t^v, z_t^v, k_t^v), 0 \le t \le T\}$ such that

(i) $y_t^v \ge h(t, x_t^v), 0 \le t \le T$;
(ii) $\{k_t^v\}$ is increasing and continuous, $k_0^v = 0$ and $\int_0^T (y_t^v - h(t, x_t^v))dk_t^v = 0$.

The objective of our problem is to look for an optimality $\bar{u} \in \mathcal{U}$ such that

$$\bar{y}_0 := y_0^{\bar{u}} = \sup_{v\in\mathcal{U}} y_0^v,$$

and we denote it by **Problem (P)**.

Remark 2.1. Problem (P) is one kind of recursive optimal control problem with obstacle constraint $y_t \ge h(t, x_t), 0 \le t \le T$. For example, if $\{x_t\}$ represents the investment value and $\{y_t\}$ represents the recursive utility in a financial market, then the constraint means that the utility is required to be higher than a given function of investment value at any time, which is quite reasonable for the rational investors. More examples for this kind of optimization problem can be referred to Wu and Yu [8].

3. Sufficient maximum principle

In this section, we shall give a sufficient maximum principle of optimality for Problem (P). The main tool is Clarke's generalized gradient defined as follows (see e.g. [10]):

Definition 3.1. Let G be a region in $\mathbb{R}^n$. Then for any locally Lipschitz function $\varphi : G \to \mathbb{R}$ and $x \in G$,

$$\partial\varphi(x) = \Big\{\xi \in \mathbb{R}^n : \langle\xi, y\rangle \le \limsup_{z\in G, z\to x, t\downarrow 0} \frac{\varphi(z + ty) - \varphi(z)}{t}, \forall y \in \mathbb{R}^n\Big\}$$

is called the Clarke's generalized gradient of φ at x.

Let us assume
(H3.1) The control domain U is a convex body in $\mathbb{R}^l$.
(H3.2) The partial derivatives of b, σ and f in (x, y) are continuous in (x, y, v).
Then we have

Theorem 3.2. *Let Assumptions (H2.1)–(H2.2) and (H3.1)–(H3.2) hold. Suppose $\bar{u}$ is an admissible control and $\{\bar{x}_t, 0 \le t \le T\}$, $\{(\bar{y}_t, \bar{z}_t, \bar{k}_t), 0 \le t \le T\}$ are the corresponding solutions of (1) and (2), respectively. Define $\tilde{h}(t, x) = h(t, x)\mathbf{1}_{\{t<T\}} + g(x)\mathbf{1}_{\{t=T\}}$, and*

$$\mathcal{H}(t, x, y, P, Q, q, v) = \langle b(t, x, v), Q\rangle + \langle\sigma(t, x, v), q\rangle + \langle f(t, x, y, v), P\rangle.$$

For a given $\mathcal{F}_t$-stopping time τ dominated by T, introduce adjoint processes $\{P_t\}$, $\{Q_t\}$ satisfying

$$\begin{cases} dP_t = f_y(t, \bar{x}_t, \bar{y}_t, \bar{u}_t)P_t dt, \quad 0 \le t \le \tau, \\ P_0 = -1, \end{cases} \tag{3}$$

and

$$\begin{cases} -dQ_t = [b_x^T(t, \bar{x}_t, \bar{u}_t)Q_t + \sigma_x^T(t, \bar{x}_t, \bar{u}_t)q_t \\ \qquad + f_x^T(t, \bar{x}_t, \bar{y}_t, \bar{u}_t)P_t]dt - q_t dW_t, \quad 0 \le t \le \tau, \\ Q_\tau = \tilde{h}_x^T(\tau, \bar{x}_\tau)P_\tau. \end{cases} \tag{4}$$

If for any $\mathcal{F}_t$-stopping time τ dominated by T, $\tilde{h}(t, \cdot)$ is concave while $\mathcal{H}(t, \cdot, \cdot, P_t, Q_t, q_t, \cdot)$ is convex, and

$$\mathcal{H}(t, \bar{x}_t, \bar{y}_t, P_t, Q_t, q_t, \bar{u}_t) = \min_{v\in U} \mathcal{H}(t, \bar{x}_t, \bar{y}_t, P_t, Q_t, q_t, v), \quad \forall v \in U, t \in [0, \tau], \tag{5}$$

then $\bar{u}$ is an optimal control of Problem (P).

Proof. For any $v \in \mathcal{U}$, set the stopping time $\tau = \inf\{0 \le t \le T : y_t^v = h(t, x_t^v)\}$. Let us consider

$$y_0^v - \bar{y}_0 = -\mathbb{E}[P_0(y_0^v - \bar{y}_0)].$$

Applying Itô's formula to $P(y^v - \bar{y})$ and $\langle Q, x^v - \bar{x}\rangle$ on the time interval $[0, \tau]$ respectively, we have

$$\begin{aligned} &\mathbb{E}[P_\tau(y_\tau^v - \bar{y}_\tau)] - \mathbb{E}[P_0(y_0^v - \bar{y}_0)] \\ &= \mathbb{E}\int_0^\tau f_y(t, \bar{x}_t, \bar{y}_t, \bar{u}_t)P_t(y_t^v - \bar{y}_t)dt \\ &\quad - \mathbb{E}\int_0^\tau P_t[f(t, x_t^v, y_t^v, v_t) - f(t, \bar{x}_t, \bar{y}_t, \bar{u}_t)]dt \\ &\quad - \mathbb{E}\int_0^\tau P_t(dk_t^v - d\bar{k}_t), \end{aligned} \tag{6}$$