Systems & Control Letters 114 (2018) 59–65

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Towards a minimal order distributed observer for linear systems

Weixin Han [a], Harry L. Trentelman [b], Zhenhua Wang [a,*], Yi Shen [a]

[a] Department of Control Science and Engineering, Harbin Institute of Technology, Harbin, 150001, PR China
[b] Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, 9700 AK Groningen, The Netherlands

ARTICLE INFO

Article history:
Received 7 August 2017
Received in revised form 9 February 2018
Accepted 20 February 2018

Keywords:
Distributed estimation
Linear system observers
Minimal order
LMI's
Sensor networks

ABSTRACT

In this paper we consider the distributed estimation problem for continuous-time linear time-invariant (LTI) systems. A single linear plant is observed by a network of local observers. Each local observer in the network has access to only part of the output of the observed system, but also receives information on the state estimates of its neighbors. Each local observer should in this way generate an estimate of the plant state. In this paper we study the problem of existence of a *reduced order* distributed observer. We show that if the observed system is observable and the network graph is a strongly connected directed graph, then a distributed observer exists with state space dimension equal to $Nn-\sum_{i=1}^{N}p_i$, where N is the number of network nodes, n is the state space dimension of the observed plant, and p_i is the rank of the output matrix of the observed output received by the ith local observer. In the case of a single observer, this result specializes to the well-known minimal order observer in classical observer design.

1. Introduction

Recently, there has been much interest in the problem of designing distributed observers for estimation of the state of a given linear time invariant plant. Whereas the classical observer problem is to find a single observer that receives the entire measured plant output in order to generate this state estimate, in the distributed version the aim is to find a given number of local observers that can communicate according to an a priori given network graph (see Fig. 1 for an illustration). Each of the local observers in the network receives only part of the plant output, but also information on the state estimates of its neighbors. Each local observer should in this way generate an estimate of the plant state. Thus, the problem of finding a distributed observer can be interpreted as the problem of finding a single observer that consists of a given number of local observers, interconnected by means of an a priori given network graph. Since each of the local observers receives only part of the plant output, properties like observability or detectability that might hold for the original plant output do no longer hold for the partial output, and hence classical observer design is not applicable for the local observer.

Among the many contributions on the distributed observer problem we mention [1,2] and [3]. In particular, in [3–5] a state augmented observer was constructed to cast the distributed estimation problem as a problem of decentralized stabilization, using the notion of fixed modes [6]. These references only discuss discrete-time systems. More recently, in [7], the idea of putting the distributed observer problem in the context of decentralized control was applied to continuous time plants. In [2,8,9] local Luenberger observers at each node were constructed, based on applying the Kalman observable decomposition. There, the observer reconstructs a certain portion of the state solely by using its own measurements, and uses consensus dynamics to estimate the unobservable portions of the state at each node. Specifically, in [1] two observer gains were designed to achieve distributed state estimation, one for local measurements and the other for the information exchange. In [10], a simple LMI based approach was proposed for the design of distributed observers.

A standard result in classical observer design states that if the plant is observable, then an observer with arbitrary fast error convergence exists of order equal to the order of the plant, say n, minus the rank of the output matrix, say p, [11]. It was argued in [12] that indeed $n-p$ is the minimal order for state observers. Of course, similarly one can address the issue of existence of a reduced, or even minimal, order distributed observer. This issue will be the topic of the present paper. We assume that our plant is a continuous-time LTI system

$$\begin{aligned}\dot{x} &= Ax\\ y &= Cx\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $y \in \mathbb{R}^m$ is the measurement output. We partition the output y as

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

* Corresponding author.
E-mail addresses: h.l.trentelman@rug.nl (H.L. Trentelman), zhenhua.wang@hit.edu.cn (Z. Wang).

https://doi.org/10.1016/j.sysconle.2018.02.011

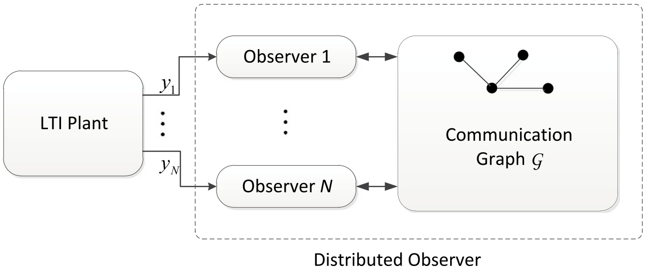

Fig. 1. Framework for distributed state estimation.

where $y_i \in \mathbb{R}^{m_i}$ and $\sum_{i=1}^{N} m_i = m$. Accordingly, we partition the output matrix as

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}$$

with $C_i \in \mathbb{R}^{m_i \times n}$. In addition, a directed graph with N nodes is given. Each node in the graph will carry a local observer. The local observer at node i has only access to the measurement $y_i = C_i x$ and to the state estimates of its neighbors, including itself. In this paper, a standing assumption will be that the communication graph is strongly connected. We will also assume that the pair (C, A) is observable. For the discrete time case, it was shown in [5] that a distributed observer of order $Nn + N - 1$ exists. This bound was re-established in [7] for continuous time plants. Again for the discrete time case, in [9] it was shown that a distributed observer exists of order Nn. Also in [1], under certain assumptions, a dynamic order Nn was shown to be sufficient. More recently, in our paper [10] we reconfirmed that for the continuous time case a dynamic order Nn suffices.

In the present paper we will improve all sufficient dynamic orders established up to now and as our main result show that, for any desired error convergence rate, a distributed observer exists of dynamic order equal to $Nn - \sum_{i=1}^{N} p_i$, where p_i is the rank of the local output matrix C_i. This result extends in a natural way the minimal order $n - p$ for a single, non-distributed observer, with p the rank of the output matrix C.

2. Preliminaries and problem formulation

2.1. Preliminaries

Notation: The rank of a given matrix M is denoted by rank M. If M has full column rank m then $M^\dagger = (M^T M)^{-1} M^T$ denotes its Moore–Penrose inverse, so $M^\dagger M = I_m$. The identity matrix of dimension N will be denoted by I_N. The vector $\mathbf{1}_N$ denotes the N-dimensional column vector comprising of all ones. For a symmetric matrix P, $P > 0\,(P < 0)$ means that P is positive (negative) definite. For a set $\{A_1, A_2, \ldots, A_N\}$ of matrices, we use $\mathrm{diag}\{A_1, A_2, \ldots, A_N\}$ to denote the block diagonal matrix with the A_i's along the diagonal, and the matrix $\begin{bmatrix} A_1^T & A_2^T & \cdots & A_N^T \end{bmatrix}^T$ is denoted by $\mathrm{col}(A_1, A_2, \ldots, A_N)$. The Kronecker product of the matrices M_1 and M_2 is denoted by $M_1 \otimes M_2$. In this paper, $\mathbb{R}^n$ will denote the n-dimensional Euclidean space. For a $p \times n$ matrix A, $\ker A := \{x \in \mathbb{R}^n \mid Ax = 0\}$ and $\mathrm{im}\, A := \{Ax \mid x \in \mathbb{R}^n\}$ will denote the kernel and image of A, respectively. If $\mathcal{V}$ is a subspace of $\mathbb{R}^n$, then $\mathcal{V}^\perp$ will denote the orthogonal complement of $\mathcal{V}$ with respect to the standard inner product in $\mathbb{R}^n$.

In this paper, a weighted directed graph is denoted by $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$, where $\mathcal{N} = \{1, 2, \ldots, N\}$ is a finite nonempty set of nodes, $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is an edge set of ordered pairs of nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ denotes the adjacency matrix. The (j, i)th entry a_{ji} is the weight associated with the edge (i, j). We have $a_{ji} \neq 0$ if and only if $(i, j) \in \mathcal{E}$. Otherwise $a_{ji} = 0$. An edge $(i, j) \in \mathcal{E}$ designates that the information flows from node i to node j. A directed path from node i_1 to i_l is a sequence of edges (i_k, i_{k+1}), $k = 1, 2, \ldots, l-1$ in the graph. A directed graph $\mathcal{G}$ is strongly connected if between any pair of distinct nodes i and j in $\mathcal{G}$, there exists a directed path from i to j, $i, j \in \mathcal{N}$.

The Laplacian $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ is defined as $\mathcal{L} := \mathcal{D} - \mathcal{A}$, where the ith diagonal entry of the diagonal matrix $\mathcal{D}$ is given by $d_i = \sum_{j=1}^{N} a_{ij}$. By construction, $\mathcal{L}$ has a zero eigenvalue with a corresponding eigenvector $\mathbf{1}_N$ (i.e., $\mathcal{L}\mathbf{1}_N = 0_N$), and if the graph is strongly connected, its algebraic multiplicity is equal to one and all the other eigenvalues lie in the open right-half complex plane.

For strongly connected graphs $\mathcal{G}$, we now review the following lemma.

Lemma 1 (*[13–15]*). *Assume $\mathcal{G}$ is a strongly connected directed graph. Then there exists a unique positive row vector $r = \begin{bmatrix} r_1, \ldots, r_N \end{bmatrix}$ such that $r\mathcal{L} = 0$ and $r\mathbf{1}_N = N$. Define $R := \mathrm{diag}\{r_1, \ldots, r_N\}$. Then $\hat{\mathcal{L}} := R\mathcal{L} + \mathcal{L}^T R$ is positive semi-definite, $\mathbf{1}_N^T \hat{\mathcal{L}} = 0$ and $\hat{\mathcal{L}}\mathbf{1}_N = 0$.*

We note that $R\mathcal{L}$ is the Laplacian of the balanced directed graph obtained by adjusting the weights in the original graph. The matrix $\hat{\mathcal{L}}$ is the Laplacian of the undirected graph obtained by taking the union of the edges and their reversed edges in this balanced digraph. This undirected graph is called the mirror of this balanced graph [13].

2.2. Problem formulation and main result

Consider the continuous-time LTI system (1), where $x \in \mathbb{R}^n$ is the state and $y \in \mathbb{R}^m$ is the measurement output. As explained in the introduction we partition the output y as $y = \mathrm{col}(y_1, \ldots, y_N)$, where $y_i \in \mathbb{R}^{m_i}$ and $\sum_{i=1}^{N} m_i = m$. Accordingly, $C = \mathrm{col}(C_1, \ldots, C_N)$ with $C_i \in \mathbb{R}^{m_i \times n}$. Here, the portion $y_i = C_i x$ is assumed to be the only output information that can be acquired by node i in the given network graph $\mathcal{G}$. The rank of the local output matrix C_i will be denoted by p_i.

In this paper, a standing assumption will be that the communication graph $\mathcal{G}$ is a strongly connected directed graph. We will also assume that the pair (C, A) is observable. However, (C_i, A) is not assumed to be observable or detectable.

We will design a distributed observer for the system (1) with the given communication network $\mathcal{G}$. The distributed observer will consist of N local observers, and the local observer at node i will have dynamics of the following form:

$$\begin{aligned} \dot{z}_i &= N_i z_i + L_i y_i + \gamma r_i M_i \sum_{j=1}^{N} a_{ij}(\hat{x}_j - \hat{x}_i) \\ \hat{x}_i &= P_i z_i + Q_i y_i \end{aligned} \tag{2}$$

where $i \in \mathcal{N}$, $z_i \in \mathbb{R}^{n-p_i}$ is the state of the local observer, $\hat{x}_i \in \mathbb{R}^n$ is the estimate of plant state at node i, a_{ij} is the (i, j)th entry of the adjacency matrix $\mathcal{A}$ of the given network, r_i is defined as in Lemma 1, $\gamma \in \mathbb{R}$ is a coupling gain to be designed, $N_i \in \mathbb{R}^{(n-p_i)\times(n-p_i)}$, $L_i \in \mathbb{R}^{(n-p_i)\times m_i}$, $M_i \in \mathbb{R}^{(n-p_i)\times n}$, $P_i \in \mathbb{R}^{n \times (n-p_i)}$ and $Q_i \in \mathbb{R}^{n \times m_i}$ are gain matrices to be designed.

The objective of distributed state estimation is to design a network of local observers (2) that cooperatively estimate the state of the plant (1). Such network of local observers is said to achieve *omniscience asymptotically*, defined as follows:

Definition 2 (*[5]*). A distributed observer (2) is said to achieve *omniscience asymptotically* if for all initial conditions on (1) and (2) we have

$$\lim_{t \to \infty} \left(\hat{x}_i(t) - x(t) \right) = 0 \tag{3}$$

for all $i \in \mathcal{N}$, i.e. the state estimate maintained by each node asymptotically converges to the true state of the plant.