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Controllability and stabilizability of multi-rate sampled data systems

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ABSTRACT

This paper gives sufficient conditions for controllability and stabilizability of multi-rate sampled data systems. For this purpose, it is firstly proposed that if the actuator communication sequence of a bandwidth limited networked system is *equidistant*, then the resulting model is mathematically equivalent to a multi-rate sampled data system. Secondly, recent results on controllability and stabilizability of bandwidth limited networked systems are reduced to the case of *equidistant* communication sequences. The established conditions enhance existing results on controllability and stabilizability of multi-rate sampled data systems. Also, it is shown that two independently established conditions for controllability and stabilizability of networked systems are equivalent in the special case of *equidistant* actuator communication sequences.

1. Introduction

Multi-rate Sampled data Control Systems (MSCS) have been subject of theoretical studies for many years [1]. Networked Control Systems are also increasingly used in several industrial applications [2] and raised new interesting topics in the intersection of network and control theories [3]. Structural properties and stabilization of Bandwidth Limited Networked Control Systems (BLNCS) have been studied during the last decade [4–10]. In [11] it is shown that the existence of constant, integer and finite delays will not compromise controllability of a BLNCS. Therefore, the results of this paper also cover the more generic case of networked control systems with bandwidth limitations and constant delays. It is proposed [10] that MSCSs can be modeled as a special class of BLNCS. In fact, the two systems are mathematically equivalent if the actuator communication sequence is *equidistant* as defined in the sequel. Results from networked control system theory are then used to enhance the existing results on MSCS. It is shown that a sufficient condition stated in [9,10] always holds and therefore, the conditions are less restrictive. Recently, two different sufficient conditions are provided for controllability and stabilizability of BLNCS by [8,9]. In the following, it is shown that these two conditions are indeed equivalent in the special case of *equidistant* actuator communication sequences.

The paper is organized in three sections. In Section 2, the main results are given and Section 3 concludes the paper.

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2. Controllability and stabilizability of multi-rate sampled data systems

A BLNCS (as well as a MSCS) is defined by (1) amended by the input Zero Order Hold (ZOH) equation (2) as follows:

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

$$u(k) = S_A(k)\,\hat{u}(k) + (I - S_A(k))\,u(k-1) \tag{2}$$

Vectors $u(k) \in R^{n_u}$ and $x(k) \in R^n$ define the control input and the state variable. Matrices $A_{n\times n}$ and $B_{n\times n_u}$ are the state and input distribution matrices of (1). Vector $\hat{u}(k) \in R^{n_u}$ represents the control input of either a BLNCS or a MSCS. Matrix $S_A(k)$ is either the scheduling matrix, modeling the bandwidth limitations of a BLNCS or a multi-rate sampling matrix modeling different sampling rates for different inputs in a MSCS. In the both cases the matrix $S_A(k)$ is defined as follows:

$$S_A(k) \triangleq diag\,(S_{Aii}(k)), \quad i = 1, \ldots, n_u \tag{3}$$

where $S_{Aii}(k)$ is defined as "1" if the ith actuator is updated at time tick k and is "0" otherwise. The multi-rate sampled data representation is obtained via updating the ith actuator, $i = 1 \ldots n_u$, each h_i seconds. The basic update time: h_{GCD} is defined as the Greatest Common Divisor (GCD) of all h_is. i.e.

$$h_{GCD} \triangleq GCD(h_1, \ldots, h_{n_u}). \tag{4}$$

Vector $\hat{u}(k)$ is the control input updated each h_{GCD} seconds. The vector $u(k) = \left[u_1^T(k) \;\ldots\; u_{n_u}^T(k)\right]^T$ represents the actual input in which $u_i(k) \in R$ is updated each h_i seconds. When a ZOH mechanism is used, each scalar input $u_i(k)$ is kept constant between any two consecutive updates. Therefore, both systems (a BLNCS with ZOH input mechanism and a MSCS) could be well represented via the same model. (See [7,8,10] for BLNCS modeling and [1] for MSCS modeling.)

Definition 1 (*[9]*)**.** A periodic actuator communication sequence (represented by σ_{pA}) is a sequence according to which the network bandwidth is arbitrated among actuators. i.e.:

$$\sigma_{pA} \triangleq (S_A(0), \dots, S_A(p-1)), \quad S_A(k) = S_A(k+p), \quad k = 0, \dots, p-1. \tag{5}$$

In this paper, the notion of actuator communication sequence is used for MSCS as well as BLNCS as in [1]. In the sequel a scalar: namely $n^{(i)}$ is attributed to each actuator in an actuator communication sequence. $n^{(i)}$ represents the number of accesses granted to that actuator during each period of the actuator communication sequence. i.e.:

$$n^{(i)} \triangleq \sum_{k=0}^{p-1} S_{Aii}(k). \tag{6}$$

Definition 2. The ith *input update index set* $\psi^{(i)}$, is a set defined for the ith actuator, describing the time indices in which access is granted to that actuator. i.e.:

$$\psi^{(i)} = \left\{a_1^{(i)}, \dots, a_{n^{(i)}}^{(i)}\right\} \triangleq \{j | S_{Aii}(j) = 1, j = 0 \dots p-1\}; \quad i = 1 \dots n_u \tag{7}$$

where $a_1^{(i)}$ is the first (smallest) and $a_{n^{(i)}}^{(i)}$ is the last (largest) time indices when the ith actuator is updated. Note that:

$$0 < a_1^{(i)} < \dots < a_{n^{(i)}}^{(i)} < p. \tag{8}$$

Definition 3. The ith *input update distance set* $\Phi^{(i)}$, is a set defined for the ith actuator, describing the difference between two consecutive access indices of that actuator.[1] i.e.:

$$\Phi^{(i)} = \left\{d_1^{(i)}, \dots, d_{n^{(i)}}^{(i)}\right\} \triangleq \left\{a_{l+1}^{(i)} - a_l^{(i)} | a_l^{(i)} \in \psi^{(i)}, \; l = 1, \dots, n^{(i)} - 1\right\} \cup \left\{p + a_1^{(i)} - a_{n^{(i)}}^{(i)}\right\}; i = 1 \dots n_u. \tag{9}$$

Definition 4. A communication sequence is called *equidistant* if all input update distances d^i for each actuator are equal. i.e.:

$$d_1^{(i)} = d_2^{(i)} = \cdots d_{n^{(i)}}^{(i)} = d_i; \quad i = 1 \dots n_u$$

in which d_i is the (constant) difference between any two consecutive update indices of the ith actuator.

Unlike MSCS where it is conventionally assumed that sampling intervals are constant, the sampling interval in a networked system is generally time varying. Therefore, the time interval between two consecutive updates of an input in MSCS is constant while it may be time varying in a general networked system. As this paper concerns *equidistant* sequences, the term "multi-rate system" may be used to represent either a BLNCS with an *equidistant* communication sequence or a MSCS.

Lemma 1. *The following holds for any* equidistant *communication sequence with period p.*

$$p = n^{(i)} d_i; \quad i = 1 \dots n_u \tag{10}$$

Proof. The period of the communication sequence could be stated as:

$$p = d_1^{(i)} + d_2^{(i)} \dots + d_{n^{(i)}}^{(i)} \tag{11}$$

If the sequence is *equidistant*, one obtains (10). ■

Lemma 2 (*[8]*)**.** *Assume that (1) is controllable (stabilizable) and the actuator communication sequence is admissible. The system described by (1)–(2) is controllable (stabilizable) if the following polynomial (known as communication sequence characteristic polynomial) has no zero in common with nonzero (unstable) eigenvalues of the system (1).*

$$g(\mu) \triangleq \det(G(\mu)) \tag{12}$$

where:

$$G(\mu) \triangleq \sum_{l=0}^{p-1} S_A(l) \sum_{j=0}^{p-1} \mu^j \prod_{q=l+1}^{p-1-j+l} \bar{S}_A(q). \tag{13}$$

It could be easily verified that $G(\mu)$ is a diagonal matrix. Therefore:

$$g(\mu) = \prod_{i=1}^{n_u} G_{ii}(\mu)$$

where:

$$G(\mu) = diag\,(G_{ii}(\mu))$$

$$G_{ii}(\mu) = \sum_{l=0}^{p-1} S_{Aii}(l) \sum_{j=0}^{p-1} \mu^j \prod_{q=l+1}^{p-1-j+l} \bar{S}_{Aii}(q). \tag{14}$$

This establishes the following corollary.

Corollary 1. *Assume that (1) is controllable (stabilizable) and the actuator communication sequence is admissible. The system described by (1)–(2) is controllable (stabilizable) if none of $G_{ii}(\mu)$, $i = 1..n_u$ has zero(s) common with nonzero eigenvalues of the system (1).*

Note 1. It is shown [11] that condition (14) remains valid after adding a constant non-negative input delay to (1)–(2). Therefore, (14) can be used for BLNCS with or without constant delays.

Theorem 1. *Assume that (1) is controllable (stabilizable) and the actuator communication sequence is admissible. The system described by (1)–(2) is controllable (stabilizable) with an* equidistant *communication sequence if the system (1) has no eigenvalue common with the zeros of the following polynomials:*

$$\sum_{j=0}^{d_i-1} \mu^j, i = 1 \dots n_u \tag{15}$$

where each d_i is the update distance of the ith actuator.

Proof. Rewrite $G_{ii}(\mu)$ as:

$$G_{ii}(\mu) = \sum_{j=0}^{p-1} n_j^{(i)} \mu^j$$

where:

$$n_j^{(i)} = \sum_{l=0}^{p-1} S_{Aii}(l) \prod_{q=l+1}^{p-1-j+l} \bar{S}_{Aii}(q).$$

Scalar $n_j^{(i)}$, counts the number of updates of the ith actuator after which there is no update during the next $p-1-j$ time slots. For *equidistant* sequences (equivalently in MSCS), two cases can be distinguished.

[1] The difference between the last update index of an actuator, namely: $a_{n^{(i)}}^{(i)}$ and the first update index in the next period should also be counted.