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On topological obstructions to global stabilization of an inverted pendulum

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ABSTRACT

We consider a classical problem of control of an inverted pendulum by means of a horizontal motion of its pivot point. We suppose that the control law can be non-autonomous and non-periodic w.r.t. the position of the pendulum. It is shown that global stabilization of the vertical upward position of the pendulum cannot be obtained for any Lipschitz control law, provided some natural assumptions. Moreover, we show that there always exists a solution separated from the vertical position and along which the pendulum never becomes horizontal. Hence, we also prove that global stabilization cannot be obtained in the system where the pendulum can impact the horizontal plane (for any mechanical model of impact). Similar results are presented for several analogous systems: a pendulum on a cart, a spherical pendulum, and a pendulum with an additional torque control.

1. Introduction

One and the same property of a system, considered in different contexts, can both be useful, and appear as an undesirable limitation: possible stability of an inverted pendulum to arbitrary horizontal movements of its pivot point [1,2] turns out to be related to the impossibility of global stabilization of a given position or motion of the pendulum.

The problem of stabilization of the vertical upward position of an inverted pendulum (or of an inverted pendulum on a cart) by means of a horizontal motion on its pivot point (or by a horizontal force, correspondingly) is a well-known problem and has been considered by many authors (see, e.g., [3–16]). This is, among other things, due to the possible applications in real-life systems [17–21].

It was proved [22] that if the configuration space of a control system has non-trivial topology, then the system cannot have a globally asymptotically stable equilibrium. To be more precise, if the configuration space is closed (compact without boundary), then global stabilization cannot be obtained. One can compare this result with the situation when relatively complex topology of the configuration space leads to non-integrability of a Hamiltonian system [23]. For instance, since the configuration space of the spherical pendulum is $\mathbb{S}^2$, the problem of global stabilization of the controlled spherical pendulum can be solved only by means of a non-continuous control [7].

For the system 'pendulum on a cart' (its phase space is $\mathbb{S} \times \mathbb{R}^3$), it is also impossible to find such a continuous control that the system would have a globally asymptotically stable equilibrium position [8,24,22]. Even the problem of stabilization of the vertical position of a one-degree-of-freedom simple inverted pendulum does not allow continuous autonomous control which would asymptotically lead the pendulum to the vertical from any initial position. This follows from the fact that a continuous function on a circle, which takes values of opposite sign, has at least two zeros, i.e., the system has at least two equilibria (see system (1)).

The following questions naturally arise. First, do the above statements remain true if we consider the pendulum only in the positions where its mass point is above the pivot point (often there exists a physical constraint in the system which does not allow the rod to be below the plane of support and it is meaningless to consider the pendulum in such positions). Second, is it true that global stabilization cannot be obtained when the control law is a time-dependent function and it is also a non-periodic function of the position of the pendulum? For a relatively broad class of problems, which may appear in practice, we show that for the both questions the answers are positive.

The main results of the paper can be described in the following way. For all systems considered in the paper it was shown [22] that they do not possess a globally asymptotically stable equilibrium and this result follows from the fact that a closed manifold cannot be contractible. At the same time, if we restrict our consideration to a contractible subset of the configuration space of the system, then there exists a vector field with a unique asymptotically stable equilibrium. However, due to limitations caused by the realization of the control mechanism, in real systems we cannot arbitrarily choose the right-hand side of the control system. In particular, we show that for the inverted pendulum there exists a contractible subset of the configuration space such that the vertical upward

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position belongs to this set, yet this equilibrium is never a global attractor. The existence of such a set is a consequence of our method of control — we try to stabilize the rod by means of a horizontal motion of the pivot point.

To be more precise, we prove that there exists a solution that does not tend to the equilibrium and the rod never becomes horizontal along it. Note that this is a solution of the system without any additional constraints. Such systems have been considered previously by many authors (see, for instance [3,6,9–11]). Let us now suppose that the pivot point of our pendulum is moving on a horizontal plane of support, i.e., the rod is constrained not to be below the horizon. The above mentioned solution still remains in the constrained system, regardless of the model of the rod-plane impact interaction. Therefore, we can claim that global stabilization cannot be obtained for the constrained system, possibly non-continuous.

The proofs are illustrative and based on the Ważewski method [25,26] and similar to the ones in [1,2,27], where the following system has been studied. Let us consider an inverted pendulum in a gravitational field with its pivot point moving along a horizontal line according to a given law of motion. It was proved that, for an arbitrary smooth function, which describes the motion of the pivot point, there always exists a solution such that the pendulum never becomes horizontal along it (never falls). If the law of motion of the pivot point is periodic, then there exists a periodic solution without falling. We add that similar results can be obtained by means of the variational approach [28].

The paper contains two main sections. In one section we consider in detail the case of control of a simple inverted pendulum (system with one degree of freedom), in another section we consider the controlled spherical pendulum and the pendulum on a cart and also present results on the impossibility of global stabilization.

2. Simple inverted pendulum

Consider the following control system:

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= u(q,p,t)\cdot \sin q - \cos q. \end{aligned} \tag{1}$$

Here and below $u \in \mathrm{Lip}(\mathbb{R}^3, \mathbb{R})$ is the control of the system defined by some locally Lipschitz function from $\mathbb{R}^3$ to $\mathbb{R}$. System (1) describes the motion of a pendulum when the acceleration of its pivot point is given by the function u. The coordinate is chosen so that $q = 0$ and $q = \pi$ correspond to the horizontal positions of the rod, $q = \pi/2$ corresponds to its vertical upward position. Without loss of generality, we assume that the mass of the pendulum, its length and the gravity acceleration equal 1. Note that we do not assume that u is periodic in q.

Suppose that we are looking for a control that would stabilize system (1) in a vicinity of a certain equilibrium position in the following sense. Let M be a subset of the phase space of the system such that the points of M correspond to the positions of the pendulum in which its rod is above the horizontal line (in our case, $M = \{0 < q < \pi\}$) and $\mu \in M$ is the equilibrium for a given control u. We assume that the control function u is chosen in such a way that there exists a closed subset $U \subset M$, $\mu \in U \setminus \partial U$ and a C^1-function $V : U \to \mathbb{R}$) with the following properties:

L1. $V(\mu) = 0$ and $V > 0$ in $U \setminus \mu$.
L2. Derivative $\dot{V}$ with respect to system (1) is negative in $U \setminus \mu$ for all t.

Since the function V can be considered as a Lyapunov function for our system, the equilibrium μ is stable. If the following (stronger) condition holds

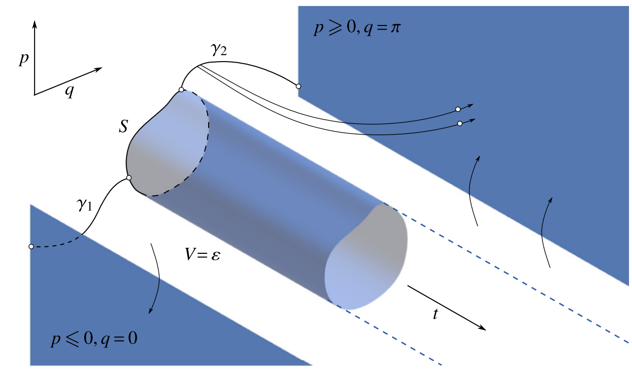

Fig. 1. Exit sets for $(M \setminus B) \times \mathbb{R}^+$. Solutions are externally tangent to $M \times \mathbb{R}^+$ at the points where $p = 0$.

L3. $\dot{V}(x, t) \leqslant -W(x) < 0$ in $U \setminus \mu$ for all t and $V(0, t) = W(0) = 0$, where $W \in C(U, \mathbb{R})$,

then μ is asymptotically stable. For instance, such a function exists in the following case. Suppose that for a given u, system (1) can be written as follows in a vicinity of μ

$$\dot{x} = Ax + f(x, t),$$

where $x = (q, p)$, A is a constant matrix and its eigenvalues have negative real parts, f is a continuous function and $f(t, x) = o(\|x\|)$ uniformly in t. Then there exists [29] a function V satisfying properties L1, L3.

We now show that in this case the control cannot be global. To be more precise, the following proposition holds.

Theorem 2.1. *Let $u(q, p, t) \in \mathrm{Lip}(\mathbb{R}^3, \mathbb{R})$ be a given control function, $\mu \in M$ be an equilibrium for system (1) and $t_0 \in \mathbb{R}$. Suppose there exists a Lyapunov function V satisfying L1 and L2, then there exists an initial condition (q_0, p_0) for $t = t_0$ and an open neighborhood $B \subset M$ of μ such that, on the interval of existence, the solution $(q(t, q_0, p_0), p(t, q_0, p_0))$ remains in $M \setminus B$.*

Proof. For any C^1 function f from $\mathbb{R}^n$ to $\mathbb{R}$ such that $f > 0$ everywhere except one point (where $f = 0$), any level set $f = \varepsilon$, for small $\varepsilon > 0$, is a homotopy sphere [30], and hence a sphere $\mathbb{S}^{n-1}$.

In our case, for small $\varepsilon > 0$, the set $V = \varepsilon$ is a circle (topologically) in the phase space. We shall denote it by S and the corresponding ball by B.

Let us consider a curve γ_1 in the phase space which connects S with the set $\{q = 0, p < 0\}$. Similarly, let γ_2 be a curve connecting S with the set $\{q = \pi, p > 0\}$ and $\gamma_1 \cap \gamma_2 = \varnothing$ (Fig. 1). Any solution starting in $M \setminus B$ at moment t_0 can leave the set $(M \setminus B) \times \mathbb{R}^+$ only through one of the following sets of the extended phase space: $S \times \mathbb{R}^+$, $\{q = 0, p \leqslant 0\} \times \mathbb{R}^+$ or $\{q = \pi, p \geqslant 0\} \times \mathbb{R}^+$. Here by $\mathbb{R}^+$ we denote the set $\{t \geqslant t_0\} \subset \mathbb{R}$.

Suppose that all solutions starting in $(S \cup \gamma_1 \cup \gamma_2) \times \{t_0\}$ leave $(M \setminus B) \times \mathbb{R}^+$. If it is true, then for every point $(q, p, t_0) \in (S \cup \gamma_1 \cup \gamma_2) \times \{t_0\}$ there is the point of first exit of the corresponding solution from $(M \setminus B) \times \mathbb{R}^+$. This point belongs to one of the above three sets (Fig. 1). Therefore, we have a map σ from the set $(S \cup \gamma_1 \cup \gamma_2) \times \{t_0\}$ to the exit set of $(M \setminus B) \times \mathbb{R}^+$. Note that $\sigma = \mathrm{id}$ on $S \times \{t_0\} \cup (\gamma_1 \cap \{q = 0, p < 0\}) \times \{t_0\} \cup (\gamma_2 \cap \{q = \pi, p > 0\}) \times \{t_0\}$, i.e. for any point (q_0, p_0, t_0) that belongs to this set, we have $\sigma(q_0, p_0, t_0) = (q_0, p_0, t_0)$. When $(q_0, p_0, t_0) \in S$, it follows from the definition of S. For the sets where $q = 0, p < 0$ and $q = \pi$, $p > 0$ it immediately follows from the first equation of system (1).