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Performance of leader–follower multi-agent systems in directed networks

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ABSTRACT

We consider leader–follower multi-agent systems in which the leader executes the desired trajectory and the followers implement the consensus algorithm subject to stochastic disturbances. The performance of the leader–follower systems is quantified by the steady-state variance of the deviation of the followers from the desired trajectory. We study the asymptotic scaling of the variance in directed lattices in one, two, and three dimensions. We show that in 1D and 2D the variance of the followers' deviation increases to infinity as one moves away from the leader, while in 3D the variance remains bounded regardless of the network size. We prove that the variance of the followers scales as a square-root function of the distance to the leader in 1D and a logarithmic function in 2D lattices.

1. Introduction

A leader–follower multi-agent system consists of a leader, who provides the desired trajectory of the multi-agent system, and a set of followers, who update their states using local relative feedback. This control strategy has a variety of applications including formation of unmanned air vehicles, control of rigid robotic bodies, and distributed estimation in sensor networks [1–11].

A fundamental question concerning the performance of the leader–follower strategy is how well the followers are able to track the trajectory of the leader when they are subject to stochastic disturbances. In large networks, the asymptotic scaling of the variance of followers' deviation from the desired trajectory is determined by the network architecture. In this paper, we focus on directed lattices in one, two, and three dimensions. We show that as one moves away from the leader, the variance of the followers increases unboundedly in 1D and 2D, whereas in 3D the variance of the followers is bounded above by a constant that is independent of the number of followers. These results resemble the performance limitation of distributed consensus in undirected tori [8]. For *directed networks*, our results for the asymptotic scaling of the performance appear to be among the first in the literature.

Our contributions are twofold. First, we obtain analytical expressions for the steady-state variance of the deviation of the followers from the leader. These expressions allow us to study the distribution of variance in leader–follower systems with directed lattices as the controller architecture. Second, we characterize the asymptotic scaling trends of the variance of the followers in 1D, 2D, and 3D directed lattices. We show that in 1D and 2D the variance of the followers scales asymptotically as a square-root function and a logarithmic function, respectively, and in 3D the variance remains bounded regardless of the network size.

This paper is organized as follows. In Section 2, we present our main results for the performance of leader–follower systems on directed lattices. We also discuss connection between our results and random walks on undirected lattices. In Section 3 we summarize our findings.

2. Leader–follower multi-agent systems on directed lattices

We consider the performance of leader–follower systems on directed lattices. By exploiting the lower triangular Toeplitz structure of the modified Laplacian matrices, we obtain analytical expressions for the variance of followers and establish its asymptotic scaling trends in large networks.

2.1. 1D lattice

Consider a set of N agents whose dynamics are modeled by the single integrators

$$\dot{\bar{x}}_n(t) = \bar{u}_n(t) + \bar{d}_n(t), \quad n = 1, \ldots, N,$$

where $\bar{x}_n(t)$ denotes the position of the nth agent, $\bar{u}_n(t)$ is the control input, and $\bar{d}_n(t)$ is a zero-mean, unit-variance stochastic disturbance. The control objective is to maintain a specified inter-agent distance, δ, and to move the formation at a desired speed, v_d.

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A leader, indexed by 0, is assumed to execute the desired trajectory at all times, that is, $\bar{x}_0(t) = v_d t$. The desired position of the nth follower is given by $\bar{x}_{n,d}(t) = v_d t + n\delta$. Each follower controls its position using the relative error from the agent ahead,

$$\bar{u}_n(t) = -(\bar{x}_n(t) - \bar{x}_{n-1}(t) - \delta) + v_d.$$

In the deviation variables $x_n(t) := \bar{x}_n(t) - \bar{x}_{n,d}(t)$, $u_n(t) := \bar{u}_n(t) - v_d$, and $d(t) := \bar{d}(t)$, the followers implement the consensus algorithm. Namely, each follower updates its state information using the relative differences between itself and its neighbor (see Fig. 1):

$$\dot{x}_n(t) = -(x_n(t) - x_{n-1}(t)) + d_n(t), \quad n = 1, \ldots, N.$$

Since the leader does not deviate from the desired trajectory, we have $x_0(t) = 0$, and $\dot{x}_0(t) = 0$. In literature [7,11,12], this is referred to as noise-free leaders, as opposed to noise-corrupted leaders whose states may deviate from the desired trajectory [6,12]. In this paper, we focus on the setup of noise-free leaders, that is, $x_0(t) = 0$.

We assume that the first follower has access to the state of the leader,

$$\dot{x}_1(t) = -x_1(t) + d_1(t).$$

By stacking the states of all followers into a vector, $x(t) = [\,x_1(t) \ldots x_N(t)\,]^T \in \mathbb{R}^N$, the state–space representation of the leader–follower system is given by

$$\dot{x}(t) = -L_1 x(t) + d(t), \tag{1}$$

where $L_1 \in \mathbb{R}^{N\times N}$ is the modified Laplacian matrix of the 1D lattice. In particular, L_1 is lower triangular Toeplitz with 1 on the main diagonal, -1 on the first subdiagonal, and zero everywhere else:

$$L_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & -1 & 1 \end{bmatrix}. \tag{2}$$

When the disturbance, $d(t) = [\,d_1(t) \ldots d_N(t)]^T \in \mathbb{R}^N$, is absent, the deviation of the followers asymptotically converges to zero. In other words, the followers converge to the desired trajectory, that is, the trajectory of the leader. In the presence of the disturbance, however, the followers converge to the desired state in the mean value. The steady-state variance of the followers can be used to quantify the deviation from the desired state:

$$V_n := \lim_{t\to\infty} E\{x_n^2(t)\}, \quad n = 1, \ldots, N,$$

where $E\{\cdot\}$ denotes the expectation operator.

We are interested in the scaling trend of the variance distribution as one moves away from the leader. Intuitively, the followers who are farther away from the leader have larger steady-state variance. It turns out that the variance of the followers increases as a square-root function of the number of followers in 1D lattices. This result is detailed in Lemma 1.

Lemma 1. *The steady-state variance of the nth follower in the 1D lattice (1) is given by*

$$V_n = \sum_{i=1}^{n} \frac{(2i-2)!}{2 \cdot 2^{2i-2}((i-1)!)^2} = \frac{n\,(2n)!}{2^{2n}\, n!\, n!}, \quad n = 1, \ldots, N. \tag{3}$$

Furthermore, V_n scales as a square-root function of n,

$$\lim_{n\to\infty} \frac{V_n}{\sqrt{n}} = \sqrt{\frac{1}{\pi}}.$$

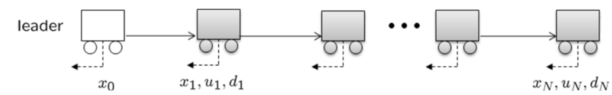

Fig. 1. The leader–follower system in 1D lattice.

To put Lemma 1 in context, recall that the variance of the undirected 1D lattice scales as a linear function of n; see e.g., [9,10]. This implies that the control architecture with directed networks outperforms the undirected counterpart in 1D lattices.

Proof. We begin with the steady-state covariance matrix

$$P := \lim_{t\to\infty} E\{x(t)x^T(t)\} = \int_0^\infty \mathrm{e}^{-L_1 t}\mathrm{e}^{-L_1^T t}\mathrm{d}t. \tag{4}$$

We compute the matrix exponential by using the inverse Laplace transform $\mathrm{e}^{-L_1 t} = \mathcal{L}^{-1}\{(sI + L_1)^{-1}\}$. Since L_1 is a lower triangular Toeplitz matrix (see (2)), it follows that $(sI + L_1)^{-1}$ is also lower triangular Toeplitz

$$(sI + L_1)^{-1} \sim \begin{bmatrix} (s+1)^{-1} & 0 & 0 \\ (s+1)^{-2} & (s+1)^{-1} & 0 \\ (s+1)^{-3} & (s+1)^{-2} & (s+1)^{-1} \end{bmatrix}.$$

In particular, $(s+1)^{-i}$ is the ith entry of the first column. By using the formula for the inverse Laplace transform

$$\mathcal{L}^{-1}\{(s+1)^{-i}\} = \frac{t^{i-1}}{(i-1)!}\mathrm{e}^{-t}, \quad i = 1, \ldots, n,$$

we obtain the nth diagonal element of the matrix $\mathrm{e}^{-L_1 t}\mathrm{e}^{-L_1^T t}$:

$$\left(\mathrm{e}^{-L_1 t}\mathrm{e}^{-L_1^T t}\right)_n = \sum_{i=1}^{n} \left(\frac{t^{i-1}}{(i-1)!}\mathrm{e}^{-t}\right)^2.$$

Performing the integration from 0 to ∞ yields

$$\begin{aligned} P_n &= \sum_{i=1}^{n} \frac{1}{((i-1)!)^2} \int_0^\infty \frac{\tau^{2(i-1)}\mathrm{e}^{-\tau}}{2^{2i-1}}\mathrm{d}\tau \\ &= \sum_{i=1}^{n} \frac{1}{((i-1)!)^2} \cdot \frac{\Gamma(2i-1)}{2^{2i-1}}, \end{aligned}$$

where we have used the change of variable $\tau = 2t$ and the formula for the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-\tau}\mathrm{d}\tau. \tag{6}$$

Since $\Gamma(z) = (z-1)!$ for positive integers, we have the desired formula (3).

To show the asymptotic scaling of P_n, we use Stirling's formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n. \tag{7}$$

Substituting into (3) with some algebra yields

$$P_n \approx \sqrt{n/\pi}. \quad \square$$

Remark 1. As a by-product of the proof, we have the following result on the total variance normalized by the number of followers

$$\Pi_N := \frac{1}{N}\sum_{n=1}^{N} V_n = \frac{(2N+1)!}{3 \cdot 2^{2N} N! N!}.$$

Using Stirling's formula (7), it can be shown that Π_N scales as a square-root function of the number of followers

$$\lim_{N\to\infty} \frac{\Pi_N}{\sqrt{N}} = \frac{2}{3\sqrt{\pi}}.$$