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Exponential state estimation, entropy and Lyapunov exponents

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ABSTRACT

In this paper we study the notion of estimation entropy established by Liberzon and Mitra. This quantity measures the smallest rate of information about the state of a system above which an exponential state estimation with a given exponent is possible. We show that this concept is closely related to the α-entropy introduced by Thieullen and we give a lower estimate in terms of Lyapunov exponents, assuming that the system preserves a volume measure, which includes all Hamiltonian and symplectic systems. Although in its current form mainly interesting from a theoretical point of view, our result could be a first step towards a more practical analysis of state estimation under communication constraints.

1. Introduction

The advent of computer-based and digitally networked control systems challenged the assumption of classical control theory that controllers and actuators have access to continuous-valued state information. This has led to massive research efforts with the aim to understand how networked systems with communication constraints between their components can be modeled and analyzed and how controllers for such systems can be designed. A foundational problem in this field is to determine the smallest rate of information above which a certain control or estimation task can be performed. There is a vast amount of literature on this topic, an overview of which is provided, e.g., in the surveys [1,2] and monographs [3–6]. Naturally, entropy concepts play a major role in describing such extremal information rates. Quantities named *topological feedback entropy*, *invariance entropy* or *stabilization entropy* have been introduced and used to describe and compute the smallest information rates for corresponding control problems (cf. [7,4,8]). These concepts are defined in a similar fashion as the well-known entropy notions in dynamical systems, such as metric or topological entropy (cf. [9,10]), and their study reveals a lot of similarities to those dynamical concepts of entropy, but sometimes also very different features.

In the problem of state estimation under communication constraints, state measurements are transmitted through a communication channel with a finite data rate to an estimator. The aim of the estimator is to build a function from the sampled measurements which approximates the real trajectory exponentially as time goes to infinity, with a given exponent. The possibility of such an estimation is crucial for many control tasks. This problem was studied in [11,5] for linear systems in a stochastic framework with the objective to bound the estimation error in probability. Here the well-known criterion (known as the *data-rate theorem*) was obtained, which states that the critical channel capacity is given by the sum of the unstable eigenvalues of the dynamical matrix. The first papers to study estimation under communication constraints for nonlinear deterministic systems were [12–16]. In [15,16], Matveev and Pogromsky studied three state estimation objectives of increasing strength for discrete-time nonlinear systems. For the weakest one, the critical bit rate was shown to be equal to the topological entropy. For the other ones, general upper and lower bounds were obtained which can be computed directly from the right-hand side of the equation generating the dynamical system. Similar studies in stochastic frameworks can be found in [17,18].

In [12–14], Liberzon and Mitra characterized the smallest bit rate for an exponential state estimation with a given exponent α for a continuous-time system on a compact subset K of its state space. As a measure for this smallest rate they introduced a quantity named *estimation entropy* $h_{\mathrm{est}}(\alpha, K)$, which coincides with the topological entropy on K when $\alpha = 0$, but for $\alpha > 0$ is no longer a purely topological quantity. Furthermore, they derived a general lower bound as well as an upper bound C of $h_{\mathrm{est}}(\alpha, K)$ in terms of α, the dimension of the state space and a Lipschitz constant of the dynamical system. They also provided an algorithm accomplishing the estimation objective with bit rate C.

The system considered in [12–14] is a flow $(\phi_t)_{t\in\mathbb{R}}$ generated by an ordinary differential equation $\dot{x} = f(x)$ on $\mathbb{R}^n$ and the initial conditions for which the state estimation is to be performed

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are constrained to a compact subset $K \subset \mathbb{R}^n$. For any exponent $\alpha \geq 0$, the estimation entropy $h_{\mathrm{est}}(\alpha, K)$ is defined similarly to the topological entropy of a continuous map on a non-compact metric space, as in Bowen [19], using (n, ε)-spanning or (n, ε)-separated sets. More precisely, the classical Bowen–Dinaburg-metrics are replaced by metrics of the form

$$d_T^{\alpha}(x, y) = \max_{0 \leq t \leq T} \mathrm{e}^{\alpha t} d(\phi_t(x), \phi_t(y)), \tag{1}$$

and the rest of the definition is completely analogous to the definition of topological entropy. Liberzon and Mitra only consider norm-induced metrics $d(x, y) = \|x - y\|$. However, if one allows more general metrics, it is easy to see that the quantity $h_{\mathrm{est}}(\alpha, K)$ depends on the choice of the metric, even in the case when K is ϕ-invariant. Hence, in contrast to the topological entropy (on a compact space), $h_{\mathrm{est}}(\alpha, K)$ is not a purely topological quantity.

The main result of this paper is based on the observation that a similar concept has been studied by Thieullen in [20], though with a completely different motivation, namely estimating the fractal dimension of compact attractors in infinite-dimensional systems. Thieullen studied the exponential asymptotic behavior of the volumes of balls $B_{\alpha}^T(x, \varepsilon)$ in the metric (1), with a volume-preserving diffeomorphism f of a compact manifold in place of the flow ϕ. His main result in [20] to some extent generalizes Pesin's formula for the metric entropy of a diffeomorphism preserving an absolutely continuous measure m, in that it expresses the exponential decay rate of $m(B_{\alpha}^T(x, \varepsilon))$ for a.e. x in terms of the Lyapunov exponents of f and the exponent α. For $\alpha = 0$, it was proved by Katok and Brin [21] that the integral over the exponential decay rate equals the metric entropy of f.

In this paper, we build a connection between the estimation entropy of Liberzon and Mitra and the α-entropy of Thieullen, using arguments from the proof of the classical variational principle for entropy as presented by Misiurewicz in [22]. To this end, we first generalize the definition of estimation entropy to discrete- and continuous-time systems on metric spaces. Then we reinterpret the estimation entropy as the topological entropy of a non-autonomous dynamical system. The time-dependency which makes the system non-autonomous enters by introducing the time-dependent metric $d_n(x, y) = \mathrm{e}^{\alpha n} d(x, y)$ on the state space, where $d(\cdot, \cdot)$ is the given metric. Then we can use ideas and results proved in [23] for general non-autonomous systems in order to provide a lower bound for the estimation entropy of a $\mathcal{C}^{1+\varepsilon}$-diffeomorphism f on a smooth compact manifold, preserving an absolutely continuous measure μ with a density that is bounded and bounded away from zero. Namely, the estimation entropy of f with respect to the exponent α is lower-bounded by the integral over the exponential decay rate of $\mu(B_{\alpha}^n(x, \varepsilon))$, expressed in terms of the μ-Lyapunov exponents and the exponent α, using Thieullen's result.

We first introduce the notions of estimation and topological entropy in Section 2. Section 3 contains the main results, in particular the lower bound Theorem 4. Section 4 provides examples and a related discussion of the practicality of Theorem 4 from an applied point of view. In Section 5, we end with some concluding remarks. A technical part of the proof and a review of the Multiplicative Ergodic Theorem are shifted to the Appendix.

2. Preliminaries

Notation: We write $\#S$ for the cardinality of a set S. The open ball of radius $\varepsilon > 0$ centered at a point x in a metric space X is denoted by $B(x, \varepsilon)$. The diameter of a subset $A \subset X$ is $\mathrm{diam}\, A := \sup_{x,y \in A} d(x, y)$. The distance from a point $x \in X$ to a set $A \subset X$ is defined by $\mathrm{dist}(x, A) := \inf_{a \in A} d(x, a)$. If $\mathbb{T} \subset \mathbb{R}$, we write $\mathbb{T}_{\geq 0} = \{t \in \mathbb{T} : t \geq 0\}$ and $\mathbb{T}_{>0} = \{t \in \mathbb{T} : t > 0\}$. If $\mathcal{U}$ is an open cover of a compact metric space (X, d), we write $L(\mathcal{U})$ for the Lebesgue number of $\mathcal{U}$, i.e., the greatest $\varepsilon > 0$ such that every ball of radius ε is contained in an element of $\mathcal{U}$. The join of open covers $\mathcal{U}_1, \ldots, \mathcal{U}_n$, denoted by $\bigvee_{i=1}^n \mathcal{U}_i$, is the open cover that consists of all intersections $U_1 \cap \ldots \cap U_n$ with $U_i \in \mathcal{U}_i$. If $(\Omega, \mathcal{F}, \mu)$ is a probability space and $\mathcal{P}$ a finite measurable partition of Ω, the entropy of $\mathcal{P}$ is defined by $H_{\mu}(\mathcal{P}) := -\sum_{P \in \mathcal{P}} \mu(P) \log_2 \mu(P)$. If $s \in \mathbb{R}$, then $\lceil s \rceil = \min\{k \in \mathbb{Z} : k \geq s\}$ and $s^+ = \max\{0, s\}$.

We first introduce a notion of estimation entropy generalizing the one in [12–14]. Let (X, d) be a metric space and $K \subset X$ compact. We consider a continuous (semi-) dynamical system $\phi : \mathbb{T} \times X \to X$, $(t, x) \mapsto \phi_t(x)$, where $\mathbb{T} \in \{\mathbb{Z}_{\geq 0}, \mathbb{R}_{\geq 0}\}$. All intervals are understood to be intersected with $\mathbb{T}$, e.g., $[0, n] = \{0, 1, \ldots, n\}$ if $\mathbb{T} = \mathbb{Z}_{\geq 0}$.

The *estimation entropy* $h_{\mathrm{est}}(\alpha, K) = h_{\mathrm{est}}(\alpha, K; \phi)$ for an $\alpha \geq 0$ is defined as follows. For $T \in \mathbb{T}_{>0}$, $\varepsilon > 0$, a set $\hat{X} = \{\hat{x}_i(\cdot)\}_{i=1}^n$ of functions $\hat{x}_i : [0, T] \to X$ is $(T, \varepsilon, \alpha, K)$-*approximating* if for each $x \in K$ there is $\hat{x}_i \in \hat{X}$ with

$$d(\phi_t(x), \hat{x}_i(t)) < \varepsilon \mathrm{e}^{-\alpha t} \text{ for all } t \in [0, T].$$

We write $s_{\mathrm{est}}(T, \varepsilon, \alpha, K)$ for the minimal cardinality of a $(T, \varepsilon, \alpha, K)$-approximating set and define

$$h_{\mathrm{est}}(\alpha, K) := \lim_{\varepsilon \downarrow 0} \limsup_{T \to \infty} \frac{1}{T} \log s_{\mathrm{est}}(T, \varepsilon, \alpha, K).$$

Here we use $\log = \log_2$ when $\mathbb{T} = \mathbb{Z}_{\geq 0}$ and $\log = \log_e$ when $\mathbb{T} = \mathbb{R}_{\geq 0}$. Alternatively, we can define $h_{\mathrm{est}}(\alpha, K)$ in terms of $(T, \varepsilon, \alpha, K)$-spanning sets, by allowing only trajectories of ϕ as approximating functions: a set $S \subset K$ is called $(T, \varepsilon, \alpha, K)$-*spanning* if for each $x \in K$ there is $y \in S$ with

$$d(\phi_t(x), \phi_t(y)) < \varepsilon \mathrm{e}^{-\alpha t} \text{ for all } t \in [0, T].$$

Writing $s_{\mathrm{est}}^*(T, \varepsilon, \alpha, K)$ for the minimal cardinality of such a set, one finds that

$$h_{\mathrm{est}}(\alpha, K) = \lim_{\varepsilon \downarrow 0} \limsup_{T \to \infty} \frac{1}{T} \log s_{\mathrm{est}}^*(T, \varepsilon, \alpha, K).$$

A third possible definition uses the concept of $(T, \varepsilon, \alpha, K)$-separated sets: a subset $E \subset K$ is $(T, \varepsilon, \alpha, K)$-*separated* if for each two $x, y \in E$ with $x \neq y$,

$$d(\phi_t(x), \phi_t(y)) \geq \varepsilon \mathrm{e}^{-\alpha t} \text{ for some } t \in [0, T].$$

Writing $n_{\mathrm{est}}^*(T, \varepsilon, \alpha, K)$ for the maximal cardinality of a $(T, \varepsilon, \alpha, K)$-separated set, one can show that

$$h_{\mathrm{est}}(\alpha, K) = \lim_{\varepsilon \downarrow 0} \limsup_{T \to \infty} \frac{1}{T} \log n_{\mathrm{est}}^*(T, \varepsilon, \alpha, K).$$

We omit the proof that these definitions are equivalent, since it works completely analogous to the case studied in [12–14]. Note that for each $T \in \mathbb{T}_{>0}$,

$$d_T^{\alpha}(x, y) := \max_{0 \leq t \leq T} \mathrm{e}^{\alpha t} d(\phi_t(x), \phi_t(y))$$

defines a metric on X. We write $B_{\alpha}^T(x, \varepsilon)$ for the ball of radius $\varepsilon > 0$ centered at $x \in X$ in this metric.

Next we recall the notion of topological entropy for non-autonomous dynamical systems as defined in [24,25]. A *topological non-autonomous dynamical system (NDS)* is a pair (X_{∞}, f_{∞}), where $X_{\infty} = (X_n)_{n=0}^{\infty}$ is a sequence of compact metric spaces (X_n, d_n) and $f_{\infty} = (f_n)_{n=0}^{\infty}$ is an equicontinuous sequence of maps $f_n : X_n \to X_{n+1}$. For any integers $i \geq 0$ and $n \geq 1$ we define

$$f_i^0 := \mathrm{id}_{X_i},\ f_i^n := f_{i+n-1} \circ \cdots \circ f_{i+1} \circ f_i,\ f_i^{-n} := (f_i)^{-n}.$$

If $\mathcal{U}_n$ is an open cover of X_n for each n, we define the entropy of f_{∞} w.r.t. $\mathcal{U}_{\infty} := (\mathcal{U}_n)_{n \geq 0}$ by

$$h(f_{\infty}; \mathcal{U}_{\infty}) := \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{N}\Big(\bigvee_{i=0}^{n} f_0^{-i} \mathcal{U}_i\Big),$$