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Solution to stochastic LQ control problem for Itô systems with state delay or input delay

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ABSTRACT

This paper is concerned with the linear quadratic (LQ) control problem for Itô stochastic systems with state delay or input delay. The main contribution is to give the explicit optimal LQ controllers for Itô stochastic systems with input delay and with state delay respectively. For the case of state delay, the optimal LQ controller is a linear function of the state at the current time and the past time. For the case of input delay, the optimal LQ controller is a linear function of the state and the past inputs. The key technology is to define the value function and complete the square based on the coupled differential equations.

1. Introduction

Time-delay systems have wide applications in economics, finance, networked control systems and physics [1–3]. The control problems for time-delay systems have thus received paramount attention since 1950s [4–8]. [4] proposes the well-known Smith predictor for the determined systems with single input delay. In [7], the optimal feedback controller for the systems with state delay is obtained based on a set of differential equations. Some other related work can be found in [9,10] and the references therein.

The stochastic uncertainties exist widely in finance [11,12], manual control and human operator modeling theory [13], and networked control systems [14] which are modeled by the Itô stochastic systems. [15] studies the stochastic LQ control problem when the cost weighting matrices for the state and control are allowed to be indefinite. The control for Itô stochastic has been studied extensively, see [11] and [12] for details.

The above mentioned work is mainly for the deterministic systems and the delay-free stochastic systems. [16] and [17] obtain the controllers for systems with delay by applying the solution to the Hamilton–Jacobi–Bellman equation with the results being implicit. [18] derives an explicit solution for the delayed stochastic control by assuming a specified costate structure as a priori. With this assumption, the LQR controller was obtained based on four complicated coupled differential equations. [19] considers the LQ control problem for the Itô stochastic system with input delays based on a special assumption regarding a martingale expression. In [20], the optimal control is derived for one kind of forward–backward linear quadratic stochastic control problem. More recently, [21] derives the explicit optimal controller based on a Riccati-ZXL difference equation for the discrete-time stochastic systems subject to both input delay and multiplicative noises. In [22], an analytical controller is given in terms of the modified Riccati differential equations for Itô stochastic systems with single input delay. By examining the existing literature, the explicit optimal controller for the Itô stochastic systems with the simultaneous existence of delay-free and one delay, remains challenging due to the unavailable of the separation principle. This motivates us to study the LQ control problem for the Itô stochastic systems with delay.

This paper focuses on the optimal control for the Itô stochastic systems with state delay or input delay. The contributions of this paper lie in that explicit expressions are given for the optimal LQ controller for Itô stochastic system with state delay and with input delay respectively. For the Itô stochastic system with state delay, the optimal feedback controller is shown to be a linear function of the current state and the past state with the feedback gain in terms of the coupled differential equations. For the Itô stochastic system with input delay, the optimal feedback controller, with the form of linear function of the state and the past inputs, is derived from the solution to the coupled differential equations defined in this paper. The main technique is to define the value function and complete the square.

The rest of the paper is organized as follows. Section 2 presents the solution to the Itô stochastic system with state delays. The results for the Itô stochastic system with input delays are illustrated in Section 3. Numerical examples are given in Section 4. Conclusion

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is provided in Section 5. Some proofs of the main results are given in the Appendix.

Notation: R^n denotes the family of n-dimensional vectors. A' denotes the transpose of the matrix A. $A \geq 0 (> 0)$ means that A is a positive semi-definite (positive definite) matrix. $(\Omega, \mathcal{F}, \mathcal{P}, \mathcal{F}_t|_{t\geq 0})$ is a complete stochastic basis so that $\mathcal{F}_0$ contains all P-null elements of $\mathcal{F}$, and the filtration is generated by the standard Brownian motion $\{\omega(t)\}_{t\geq 0}$. Denote E as mathematical expectation operator. The following sets are useful throughout the paper:

$$C_{[-h,0]} = \{\varphi(t) : [-h, 0] \to R^m \text{ is continuous}\},$$
$$\bar{C}_{[-h,0)} = \{\varphi(t) : [-h, 0) \to R^m \text{ is continuous and } \sup_{-h\leq t<0} \|\varphi(t)\| < \infty\},$$
$$L^2_{\mathcal{F}}(0, T; R^m) = \{\varphi(t)_{t\in[0,T]} \text{ is an } \mathcal{F}_t - \text{adapted stochastic process s.t. } E\int_0^T \|\varphi(t)\|^2 dt < \infty\}.$$

2. Itô stochastic system with state delay

2.1. Problem formulation

Consider the following Itô stochastic system with state delay:

$$\begin{aligned} dx(t) &= [A_0x(t) + A_1x(t-h) + Bu(t)]dt \\ &\quad + [\bar{A}_0x(t) + \bar{A}_1x(t-h) + \bar{B}u(t)]d\omega(t), \qquad (1) \\ x(s) &= \mu(s), \quad s \in [-h, 0], \end{aligned}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $\omega(t)$ is the one-dimensional standard Brownian motion and $h > 0$ is the state delay. $\mu(s) \in C_{[-h,0]}$, $s \in [-h, 0]$ is the initial value and $A_i, B, \bar{A}_i, \bar{B}, i = 0, 1$ are constant matrices with compatible dimensions. The set of the admissible controllers is $L^2_{\mathcal{F}}(0, T; R^m)$.

The associated quadratic cost function for system (1) is given by

$$J(T) = E\left\{\int_0^T [x'(t)Qx(t) + u'(t)Ru(t)]dt + x'(T)Hx(T)\right\} \qquad (2)$$

where $H \geq 0$, $Q \geq 0$, and $R \geq 0$ are constant matrices with compatible dimensions.

Problem 1. Find the $\mathcal{F}_t$-adapted $u(t)$ to minimize (2) subject to system (1).

2.2. Main results

To give the optimal controller, we define the coupled differential Riccati equations for $t \in (T-h, T]$, $\theta \in (t, T]$,

$$\begin{aligned} -\dot{P}(t) &= P(t)A_0 + A_0'P(t) + \bar{A}_0'P(t)\bar{A}_0 \\ &\quad + Q - L'(t)R^{-1}(t)L(t) \qquad (3) \\ -\frac{\partial P(t,\theta)}{\partial t} &= A_0'P(t,\theta) - L'(t)R^{-1}(t)L(t,\theta), \qquad (4) \\ P(t,t) &= P(t)A_1 + \bar{A}_0'P(t)\bar{A}_1 - L'(t)R^{-1}(t)L_1(t), \qquad (5) \end{aligned}$$

where

$$\begin{aligned} R(t) &= R + \bar{B}'P(t)\bar{B}, \qquad (6) \\ L(t) &= B'P(t) + \bar{B}'P(t)\bar{A}_0, \qquad (7) \\ L(t,\theta) &= B'P(t,\theta), \qquad (8) \\ L_1(t) &= \bar{B}'P(t)\bar{A}_1, \qquad (9) \end{aligned}$$

with terminal conditions $P(T) = H$.

On the other hand, for $t \in [0, T-h]$, $\theta \in (t, t+h]$, we define the following coupled differential Riccati equations,

$$\begin{aligned} -\dot{P}(t) &= P(t)A_0 + A_0'P(t) + \bar{A}_0'P(t)\bar{A}_0 + \bar{A}_1'P(t+h)\bar{A}_1 + Q \\ &\quad + P(t, t+h) + P'(t, t+h) - L'(t)R^{-1}(t)L(t) \\ &\quad - L'(t+h, t+h)R^{-1}(t+h)L(t+h, t+h), \qquad (10) \\ -\frac{\partial P(t,\theta)}{\partial t} &= A_0'P(t,\theta) + P'(\theta, t+h)A_1 \\ &\quad - L'(\theta, t+h)R^{-1}(\theta)L_1(\theta) \\ &\quad - L'(t)R^{-1}(t)L(t,\theta) \\ &\quad - \int_t^\theta L'(s, t+h)R^{-1}(s)L(s,\theta)ds, \qquad (11) \\ P(t,t) &= P(t)A_1 + \bar{A}_0'P(t)\bar{A}_1 - L'(t)R^{-1}(t)L_1(t), \qquad (12) \end{aligned}$$

where $R(t)$, $L(t)$, $L(t,\theta)$ and $L_1(t)$ are as defined in (6)–(9).

The main result of this section is now presented.

Theorem 1. *Assuming that (3)–(12) have solutions such that $R(t) > 0$, then the optimal control law that minimizes the cost function (2) is given by*

$$\begin{aligned} u(t) &= -R^{-1}(t)L(t)x(t) - R^{-1}(t)\int_t^{\min\{t+h,T\}} L(t,\theta)x(\theta-h)d\theta \\ &\quad - R^{-1}(t)L_1(t)x(t-h), \qquad (13) \end{aligned}$$

where $R(t)$, $L(t)$, $L(t,\theta)$ and $L_1(t)$ are defined as in (6)–(9).

Proof. The proof is put into Appendix A. □

Remark 1. As is known to all that the analytical solutions to the coupled differential Riccati equations (3)–(12) are unavailable, even for the standard differential Riccati equations, the analytical solution is unaccessible. One technique is to discretize the differential Riccati equations. More specifically, giving a partition: $0 = t_0 < \cdots < t_{N+1} = T$, let $\Delta = t_{k+1} - t_k$, $\Delta d = h$, and simply denote the simulation variables of $P(t_k)$, $P(t_k, \theta)$, $P(t_k, t_k)$, $R(t_k)$, $L(t_k)$ $L(t_k, \theta)$ and $L_1(t_k)$ at time t_k as P_k, P_k^{k+i}, P_k^k, R_k, L_k L_k^{k+i} and L_k^k, $i = 1, \ldots, d$. It is noted that when Δ is small enough, P_k, P_k^{k+i} approximate to the solutions of $P(t)$, $P(t,\theta)$. Accordingly, for $k \geq N-d+1$, the coupled Riccati equations (3)–(9) can be discretized as the following Riccati equations in sequence:

$$\begin{aligned} P_k &= P_{k+1} + \Delta P_{k+1}A_0 + \Delta A_0'P_{k+1} + \Delta\bar{A}_0'P_{k+1}\bar{A}_0 \\ &\quad + \Delta Q - \Delta L_k'R_k^{-1}L_k, \\ P_k^{k+i} &= P_{k+1}^{k+i} + \Delta A_0'P_{k+1}^{k+i} - \Delta L_k'R_k^{-1}L_k^{k+i}, \quad i = 1, \ldots, d, \\ P_k^k &= P_{k+1}A_1 + \bar{A}_0'P_{k+1}\bar{A}_1 - L_k'R_k^{-1}L_k^k, \end{aligned}$$

where

$$\begin{aligned} R_k &= R + \bar{B}'P_{k+1}\bar{B}, \\ L_k &= B'P_{k+1} + \bar{B}'P_{k+1}\bar{A}_0, \\ L_k^{k+i} &= B'P_{k+1}^{k+i}, \quad i = 1, \ldots, d, \\ L_k^k &= \bar{B}'P_{k+1}\bar{A}_1, \end{aligned}$$

with the terminal values $P_{N+1} = H$, $P_{N+1}^{N+j} = 0$, $j = 1, \ldots, d+1$. For $0 \leq k < N-d+1$, the coupled Riccati equations (10)–(12) can be discretized as the following Riccati equations in sequence:

$$\begin{aligned} P_k &= P_{k+1} + \Delta P_{k+1}A_0 + \Delta A_0'P_{k+1} + \Delta\bar{A}_0'P_{k+1}\bar{A}_0 + \Delta\bar{A}_1P_{k+d+1}\bar{A}_1 \\ &\quad + \Delta Q + \Delta P_{k+1}^{k+d} + \Delta(P_{k+1}^{k+d})' - L_k'R_k^{-1}L_k - (L_{k+d}^{k+d})'R_{k+d}^{-1}L_{k+d}^{k+d}, \\ P_k^{k+i} &= P_{k+1}^{k+i} + \Delta A_0'P_{k+1}^{k+i} + \Delta(P_{k+i}^{k+d})'A_1 - \Delta(L_{k+i}^{k+d})'R_{k+i}^{-1}L_{k+i}^{k+i} \\ &\quad - \Delta L_k'R_k^{-1}L_k^{k+i} - \Delta^2\sum_{j=0}^{i}(L_{k+j}^{k+d})'R_{k+j}^{-1}L_{k+j}^{k+i}, \quad i = 1, \ldots, d, \\ P_k^k &= P_{k+1}A_1 + \bar{A}_0'P_{k+1}\bar{A}_1 - L_k'R_k^{-1}L_k^k, \end{aligned}$$

where R_k, L_k, L_k^{k+i} and L_k^k are defined as above.