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Intrinsic and apparent singularities in differentially flat systems, and application to global motion planning

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ABSTRACT

In this paper, we study the singularities of differentially flat systems, in the perspective of providing global or semi-global motion planning solutions for such systems: flat outputs may fail to be globally defined, thus potentially preventing from planning trajectories leaving their domain of definition, the complement of which we call *singular*. Such singular subsets are classified into two types: *apparent* and *intrinsic*. A rigorous definition of these singularities is introduced in terms of atlas and local charts in the framework of the differential geometry of jets of infinite order and Lie–Bäcklund isomorphisms. We then give an inclusion result allowing to effectively compute all or part of the intrinsic singularities. Finally, we show how our results apply to the global motion planning of the celebrated example of non holonomic car.

1. Introduction

Differential flatness has become a central concept in non-linear control theory for the past two decades. See [1,2], the overviews [3,4] and [5] for a thoroughgoing presentation.

Consider a non-linear system on a smooth n-dimensional manifold X given by

$$\dot{x} = f(x, u) \quad (1)$$

where $x \in X$ is the n-dimensional state vector and $u \in \mathbb{R}^m$ the input or control vector, with $m \leq n$ to avoid trivial situations.

We consider infinitely prolonged coordinates of the form $(x, \overline{u}) \triangleq (x, u, \dot{u}, \ddot{u}, \dots) \in X \times \mathbb{R}^m_\infty \triangleq X \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots$ where the latter cartesian product is made of a countably infinite number of copies of $\mathbb{R}^m$.

Roughly speaking, system (1) is said to be (differentially) flat[1] at a point $(x_0, \overline{u}_0) \triangleq (x_0, u_0, \dot{u}_0, \dots) \in X \times \mathbb{R}^m_\infty$, if there exists an m-dimensional vector $y = (y_1, \dots, y_m)$ satisfying the following statements:

- y is a smooth function of x, u and time derivatives of u up to a finite order $\beta = (\beta_1, \dots, \beta_m)$, *i.e.* $y = \Psi(x, u, \dot{u}, \dots, u^{(\beta)})$, where $u^{(\beta)}$ stands for $(u_1^{(\beta_1)}, \dots, u_m^{(\beta_m)})$ and where $u_i^{(\beta_i)}$ is the β_ith order time derivative of u_i, $i = 1, \dots, m$, in a neighborhood of the point $(x_0, \overline{u}_0)$;
- y and its successive time derivatives $\dot{y}, \ddot{y}, \dots$ are locally differentially independent in this neighborhood;
- x and u are smooth functions of y and its time derivatives up to a finite order $\alpha = (\alpha_1, \dots, \alpha_m)$, *i.e.* $(x, u) = \Phi(y, \dot{y}, \dots, y^{(\alpha)})$ in a neighborhood of the point $(y_0, \dot{y}_0, \dots) \triangleq (\Psi(x_0, u_0, \dot{u}_0, \dots, u_0^{(\beta)}), \dot{\Psi}(x_0, u_0, \dot{u}_0, \dots, u_0^{(\beta+1)}), \dots)$.

Then the vector y is called *flat output*.

Note that the latter informal definition becomes rigorous if we regard the above defined functions Φ and Ψ as smooth functions over infinite order jet spaces endowed with the product topology[2] [2,5–7]. They are then called *Lie–Bäcklund isomorphisms* and are inverse one of each other (see [2,5]). However, these functions may be defined on suitable neighborhoods that need not cover

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[1] This is not a rigorous definition but rather an informal presentation, without advanced mathematics, of the flatness concept. Problems associated to this informal definition are reported in [5, Section 5.2]. For a rigorous definition, in the context of implicit systems, the reader may refer to Definitions 1 and 2 of Section 2.

[2] Recall that in this topology, a continuous function only depends on a finite number of variables, *i.e.* , in this context of jets of infinite order, on a finite number of successive derivatives of u (see *e.g.* [5, Section 5.3.2]).

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the whole space. We thus may want to know where such isomorphisms do not exist at all, a set that may be roughly qualified of *intrinsically singular*, thus motivating the present work: if two points are separated by such an intrinsic singularity, it is intuitively impossible to join them by a smooth curve satisfying the system differential equations and, thus, to globally solve the motion planning problem.[3]

More precisely, the notions of *apparent and intrinsic singularities* are introduced thanks to the construction of an *atlas*, that we call *Lie–Bäcklund atlas*, where *local charts* are made of the open sets where the Lie–Bäcklund isomorphisms, defining the flat outputs, are non degenerated, in the spirit of [8,9] where a comparable idea was applied to a quadcopter model. Intrinsic singularities are then defined as points where flat outputs fail to exist, *i.e.* that are contained in no above defined chart at all. Other types of singularities are called apparent, as they can be ruled out by switching to another flat output well defined in an intersecting chart. Our intrinsic singularity notion may be seen as a generalization of the one introduced in [10] in the particular case of two-input driftless systems such as cars with trailers, and restricted to the so-called x-flat outputs.

Our main result, apart from the above Lie–Bäcklund atlas and singularities definition, then concerns the inclusion of a remarkable and effectively computable set in the set of intrinsic singularities. Note that, since finitely computable necessary and sufficient conditions of non existence of flat output are not available in general [5,11], an easily computable complete characterization of the set of intrinsic singularities is not presently known and it may be useful to label all or part of the singularities as intrinsic thanks to their membership of another set.

To briefly describe this result, we start from the necessary and sufficient conditions for the existence of local flat outputs of meromorphic systems of [11].[4] It consists in firstly transforming the system (1) in the locally equivalent implicit form:

$$F(x,\dot{x}) = 0 \tag{2}$$

where F is assumed meromorphic, and introducing the operator τ, the trivial Cartan field on the manifold of global coordinates $(x,\dot{x},\ddot{x},\dots)$, given by $\tau = \sum_{i=1}^{n}\sum_{j\geq 0} x_i^{(j+1)}\frac{\partial}{\partial x_i^{(j)}}$. Then, we compute the *diagonal* or *Smith–Jacobson decomposition* [5,15] of the following polynomial matrix:

$$P(F) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}}\tau \tag{3}$$

a matrix that describes the variational system associated to (2), and that lies in the ring of matrices whose entries are polynomials in the operator τ with meromorphic coefficients.

We prove that the set of intrinsic singularities contains the set where $P(F)$ is *not* hyper-regular (see [5]). As a corollary, we deduce that if an equilibrium point is not first order controllable, then it is an intrinsic singularity.

These results are applied to the global motion planning problem of the well-known non-holonomic car, which is only used here as a benchmark in order to show how the classical and simple flatness-based motion planning methodology can be extended in presence of singularities. It is also meant to help the reader verifying that the introduced concepts, in the relatively arduous context of Lie–Bäcklund isomorphisms, are nevertheless intuitive and well suited to this situation.

Note that different approaches, also leading to global results, have already been extensively developed in the context of non holonomic systems, based on controllability, Lie brackets of vector fields and piecewise trajectory generation by sinusoids [16–19], or using Brockett–Coron stabilization results [20,21]. However, though some particular nonholonomic systems, as the car example, happen to be flat, our approach applies to the class of flat systems which is different, including *e.g.* pendulum systems, unmanned aerial vehicles and many others that do not belong to the nonholonomic class (see [3–5,8,9]).

Remark that, in the car example, the obtained intrinsic singularities are the same as the ones revealed in [16–19] where first order controllability fails to hold, or, according to [20,21], where stabilisation by continuous state feedback is impossible. However, the degree of generality of this coincidence is not presently known.

The paper is organized as follows. In Section 2, we introduce the basic language of Lie–Bäcklund atlas and charts. Then this leads to a computational approach for calculating intrinsic singularities. In particular, their links with the hyper-singularity of the polynomial matrix (3) of the variational system are established in Proposition 2 and Theorem 1, and then specialized in Corollary 1 to the case of equilibrium points.

In Section 3.4, we apply our results to the non holonomic car. We build an explicit Lie–Bäcklund atlas for this model, compute the set of intrinsic singularities and apply the atlas construction to trajectory planning where the route contains several apparent singularities and starts and ends at intrinsically singular points. Finally, conclusions are drawn in Section 4.

2. Lie–Bäcklund atlas, apparent and intrinsic singularities

Recall from the introduction that we consider the controlled dynamical system in explicit form (1), where x evolves in some n-dimensional manifold X. The control input u lies in $\mathbb{R}^m$. Then the system can be seen as the zero set of $\dot{x} - f(x,u)$ in $\mathrm{T}X \times \mathbb{R}^m$, where $\mathrm{T}X$ is the tangent bundle of X. From now on, we assume that the Jacobian matrix $\frac{\partial f}{\partial u}(x,u)$ has rank m for every (x,u).

Converting system (1) into its implicit form consists in eliminating the input u or, more precisely, in computing its image by the projection π from $\mathrm{T}X \times \mathbb{R}^m$ onto $\mathrm{T}X$ to get the implicit relation (2), where we assume that $F : (x,\dot{x}) \in \mathrm{T}X \mapsto \mathbb{R}^{n-m}$ is a meromorphic function, with $m \leq n$.

Following [5,11], we embed the state space associated to (2) into a diffiety (see [7]), i.e. into the manifold $\mathfrak{X} \triangleq X \times \mathbb{R}^n_\infty$, where we have denoted by $\mathbb{R}^n_\infty$ the product of a countably infinite number of copies of $\mathbb{R}^n$, with coordinates $\bar{x} \triangleq (x,\dot{x},\ddot{x},\dots,x^{(k)},\dots)$, endowed with the trivial Cartan field:

$$\tau_{\mathfrak{X}} \triangleq \sum_{i=1}^{n}\sum_{j\geq 0} x_i^{(j+1)}\frac{\partial}{\partial x_i^{(i)}}.$$

Note that $\tau_{\mathfrak{X}}$ is such that the elementary relations $\tau_{\mathfrak{X}} x^{(k)} = x^{(k+1)}$ hold for all $k \in \mathbb{N}$. The integral curves of both (1) and (2) thus belong to the zero set of $\{F, \tau_{\mathfrak{X}}^k F \mid k \in \mathbb{N}\}$ in $\mathfrak{X}$. However, there might exist points $\bar{x} = (x,\dot{x},\ddot{x},\dots,x^{(k)},\dots) \in \mathfrak{X}$ such that the fiber $\pi^{-1}(x,\dot{x})$ above $\bar{x}$ is empty, *i.e.* such that there does not exist a $u \in \mathbb{R}^m$ such that $\dot{x} - f(x,u) = 0$. We indeed naturally exclude such points. It is easily proven that the integral curves of (1) and (2) coincide on the set $\mathfrak{X}_0$ given by

$$\mathfrak{X}_0 = \{\bar{x} \in \mathfrak{X} \mid \tau_{\mathfrak{X}}^k F(\bar{x}) = 0, \forall k \in \mathbb{N}\} \setminus \{\bar{x} \in \mathfrak{X} \mid \pi^{-1}(x,\dot{x}) = \emptyset\}.$$

Therefore, the system trajectories are uniquely defined by the triple $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ that we call *the system* from now on (see [5]). Without loss of generality, we may consider that this system is restricted to $\mathfrak{X}_0$.

In order to get rid of any reference to an explicit system, such as the complementary of the empty fibers of the projection π, we

[3] By global motion planning problem, we mean that two arbitrary points of the infinite jet space associated to the system, once the set of intrinsic singularities has been removed, can be joined by a system's trajectory, and thus that this set is connected by arcs.

[4] Other approaches to flatness characterization may be found in [12–14].