Systems & Control Letters 112 (2018) 18–23

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Time-varying bang–bang property of time optimal controls for heat equation and its application ☆

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ARTICLE INFO

Article history:
Received 19 March 2017
Received in revised form 11 December 2017
Accepted 22 December 2017

Keywords:
Heat equation
Bang–bang property
Time optimal control problem
Target optimal control problem

ABSTRACT

In this paper, the bang–bang property of time optimal controls for heat equation is established. Compared with the existing results on these problems, the bound of control variables is not a constant but a time-varying function. As an application of the bang–bang property, some kind of relation between the time optimal control problem and its corresponding target optimal control problem is considered.

1. Introduction

Let T be a positive number and Ω be a nonempty, open bounded domain in $\mathbb{R}^N (N \geq 1)$, with smooth boundary $\partial\Omega$. Let ω be a nonempty and open subset of Ω. Consider the following controlled system:

$$\begin{cases} \partial_t y - \Delta y = \chi_\omega \chi_{(\tau,T)} u & \text{in } \Omega \times (0,T), \\ y = 0 & \text{on } \partial\Omega \times (0,T), \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here $y_0 \in L^2(\Omega)$ is a given function, $u \in L^\infty(0,T;L^2(\Omega))$, $\tau \in [0,T)$, χ_ω and $\chi_{(\tau,T)}$ stand for the characteristic functions of ω and (τ,T), respectively. We denote the solution of (1.1) by $y(\cdot; y_0, \chi_{(\tau,T)}u)$.

In this paper, we set

$$L^\infty_+(0,T) \triangleq \{v \in L^\infty(0,T) \big| v(t) > 0 \ \text{ a.e. } \ t \in (0,T)\}.$$

Denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and inner product of $L^2(\Omega)$, respectively, by $B(0,r)$ and $\bar{B}(0,r)$ the open and closed ball of $L^2(\Omega)$ centered in 0 and of radius $r > 0$, respectively.

The null and approximate controllability of system (1.1) has been studied in many works (see, e.g. [1,2]). Especially, for each $\varepsilon > 0$, we have $\|y(T;y_0,0)\| \leq \varepsilon$ when T is large enough. The readers can also refer to [3–9] for more discussions on controlled heat equations.

Three kinds of optimal control problems: time optimal control problem, target optimal control problem and norm optimal control problem, are important and interesting problems of optimal control theory. For the deterministic systems, the reader can refer to [10] to obtain the recent results and open problems. The readers can also refer to [11–16,8] for time optimal control problems. For the stochastic ones, norm optimal control problems were considered in [17–19] for controlled stochastic ordinary differential equations, and in [20] for controlled stochastic heat equations. The reader can also refer to [21,9,22] for the work on equivalence between these three optimal control problems.

For a given function $M(\cdot) \in L^\infty_+(0,T)$, we define

$$\mathcal{U}_{M(\cdot)} \triangleq \Big\{v \in L^\infty(0,T;L^2(\Omega)) \Big| \|v(t)\| \leq M(t) \text{ for a.e. } t \in (0,T)\Big\}, \tag{1.2}$$

$$\mathcal{R}(y_0,\tau,T) \triangleq \Big\{y(T;y_0,\chi_{(\tau,T)}u) \Big| u \in \mathcal{U}_{M(\cdot)}\Big\}, \quad \forall\, \tau \in [0,T),$$

$$\mathcal{R}(y_0,T,T) \triangleq \{y(T;y_0,0)\},$$

and

$$\tau_0 \triangleq \sup_{\tau\in[0,T)} \{\tau \big| y(T;y_0,\chi_{(\tau,T)}u) = 0, u \in \mathcal{U}_{M(\cdot)}\}. \tag{1.3}$$

☆ This work is supported in part by the National Natural Science Foundation of China (11526167, 11371375, 61673399), China Postdoctoral Science Foundation and Central South University Postdoctoral Science Foundation (2017M610503), the Fundamental Research Funds for the Central Universities (XDJK2017C054), the Natural Science Foundation of CQCSTC (2015jcyjA00017).

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https://doi.org/10.1016/j.sysconle.2017.12.008
0167-6911/

Here, if the set in the right hand side of (1.3) is empty, we agree that $\tau_0 = 0$.

In this paper, we assume that $y_0 \neq 0$, and define

$$\varepsilon_T \triangleq \|y(T; y_0, 0)\| > 0. \tag{1.4}$$

The time optimal control problem considered in this paper reads as follows:

$$\tau(\varepsilon) \triangleq \sup_{\tau\in[0,T)} \left\{\tau \middle| y(T; y_0, \chi_{(\tau,T)}u) \in \bar{B}(0, \varepsilon), u \in \mathcal{U}_{M(\cdot)}\right\}, \tag{1.5}$$

where $\varepsilon > 0$. In this problem, if $u \in \mathcal{U}_{M(\cdot)}$ and $y(T; y_0, \chi_{(\tau,T)}u) \in \bar{B}(0, \varepsilon)$ for some $\tau \in [0, T)$, we call u an admissible control; if $\tau(\varepsilon) \in [0, T)$ and $u^* \in \mathcal{U}_{M(\cdot)}$ satisfy $y(T; y_0, \chi_{(\tau(\varepsilon),T)}u^*) \in \bar{B}(0, \varepsilon)$, we call $\tau(\varepsilon)$ and $\chi_{(\tau(\varepsilon),T)}u^*$ the optimal time and a time optimal control to problem (1.5), respectively.

We now introduce the corresponding target optimal control problem, with $\tau \in [0, T)$, in the following manner:

$$\varepsilon(\tau) \triangleq \inf\left\{\|y(T; y_0, \chi_{(\tau,T)}u)\| \middle| u \in \mathcal{U}_{M(\cdot)}\right\}. \tag{1.6}$$

In this problem, u^* is called target optimal control if

$$u^* \in \mathcal{U}_{M(\cdot)} \text{ and } \|y(T; y_0, \chi_{(\tau,T)}u^*)\| = \varepsilon(\tau).$$

It is obvious that $0 \leq \varepsilon(\tau) \leq \varepsilon_T$.

The first main result of this paper is to establish the following time-varying bang–bang property to the problem (1.5):

Theorem 1.1. *Suppose that $M(\cdot) \in L^\infty_+(0, T)$, $\varepsilon > 0$, $\mathcal{R}(y_0, T, T) \cap \bar{B}(0, \varepsilon) = \emptyset$ and $\mathcal{R}(y_0, 0, T) \cap \bar{B}(0, \varepsilon) \neq \emptyset$. Then the following two conclusions are true:*

(i) The problem (1.5) is solvable, and the time optimal control $\chi_{(\tau(\varepsilon),T)}u^*$ is unique;

(ii) The time optimal control $\chi_{(\tau(\varepsilon),T)}u^*$ to the problem (1.5) satisfies the following time-varying bang–bang property:

$$\|u^*(t)\| = M(t) \text{ for a.e. } t \in (\tau(\varepsilon), T) \tag{1.7}$$

and

$$\|y(T; y_0, \chi_{(\tau(\varepsilon),T)}u^*)\| = \varepsilon. \tag{1.8}$$

The bang–bang property of time optimal control problem, first studied in [23], is a classical problem in control theory. There also exist many works on this topic (see, e.g. [24–26,16,27,28,8,29,30]). But, to the best of our knowledge, there does not exist any work on this kind of time-varying bang–bang property. As an application of Theorem 1.1, the second main result of this paper is to obtain the relation between problems (1.5) and (1.6) as follows:

Theorem 1.2. *Let $M(\cdot) \in L^\infty_+(0, T)$. Then the function $\tau \mapsto \varepsilon(\tau)$ is strictly monotonically increasing and Lipschitz continuous from (τ_0, T) onto $(\varepsilon(\tau_0), \varepsilon_T)$. Furthermore, it holds that*

$$\varepsilon = \varepsilon(\tau(\varepsilon)), \quad \forall\, \varepsilon \in (\varepsilon(\tau_0), \varepsilon_T), \tag{1.9}$$

and

$$\tau = \tau(\varepsilon(\tau)), \quad \forall\, \tau \in (\tau_0, T). \tag{1.10}$$

When $M(\cdot) \equiv M_0$ ($M_0 > 0$ is a constant), a kind of equivalence theorem of time optimal control problem and target optimal control problem has been discussed in [9].

We organize this paper as follows. In Section 2, we shall prove the existence of target optimal controls for the problem (1.6), and discuss some properties of optimal controls (see Lemma 2.1). Then we prove Theorems 1.1 and 1.2 in Section 3.

2. Existence of optimal controls for (1.6) and their properties

Lemma 2.1. *Let $\tau \in [0, T)$ be fixed. Then the following two conclusions hold:*

(i) There exists at least one time optimal control for the problem (1.6);

(ii) $\tau_0 \in [0, T)$. Moreover, if $\tau \in (\tau_0, T)$, then any target optimal control u^* to the problem (1.6) satisfies the following property:

$$\|u^*(t)\| = M(t) \text{ for a.e. } t \in (\tau, T). \tag{2.1}$$

Proof. (i) Let $\tau \in [0, T)$ be fixed. It is obvious that $\varepsilon(\tau) \geq 0$. Let $\{u_n\}_{n\geq1} \subset \mathcal{U}_{M(\cdot)}$ be a minimal sequence of (1.6), i.e.,

$$\|y(T; y_0, \chi_{(\tau,T)}u_n)\| \to \varepsilon(\tau) \text{ as } n \to \infty. \tag{2.2}$$

Since

$$\|u_n(t)\| \leq M(t) \leq \|M(\cdot)\|_{L^\infty(0,T)} \quad \text{for a.e.} \quad t \in (0, T),$$

there exists a subsequence of $\{u_n\}_{n\geq1}$, still denoted by itself, and $v^* \in L^\infty(0, T; L^2(\Omega))$ such that

$$u_n \to v^* \text{ weakly star in } L^\infty(0, T; L^2(\Omega)). \tag{2.3}$$

By (2.3) and a standard argument, there is a subsequence of $\{n\}_{n\geq1}$, still denoted in the same way, such that

$$y(\cdot; y_0, \chi_{(\tau,T)}u_n) \to y(\cdot; y_0, \chi_{(\tau,T)}v^*) \text{ strongly in } C([0, T]; L^2(\Omega)). \tag{2.4}$$

It follows from (2.2) and (2.4) that

$$\|y(T; y_0, \chi_{(\tau,T)}v^*)\| = \varepsilon(\tau). \tag{2.5}$$

We next show that

$$\|v^*(t)\| \leq M(t) \text{ a.e. } t \in (\tau, T). \tag{2.6}$$

By contradiction, there would exist $\delta_0 > 0$ and a measurable set $E_0 \subset (\tau, T)$ with $|E_0| > 0$ such that

$$\|v^*(t)\| > M(t) + \delta_0, \quad \forall\, t \in E_0, \tag{2.7}$$

where $|E_0|$ represents the Lebesgue measure of E_0. Then

$$\int_{E_0} \|v^*(t)\|\mathrm{d}t \geq \int_{E_0} M(t)\mathrm{d}t + \delta_0|E_0|. \tag{2.8}$$

Set

$$\zeta(t) \triangleq \begin{cases} 0, & t \in (0, T) \setminus E_0, \\ \dfrac{v^*(t)}{\|v^*(t)\|}, & t \in E_0. \end{cases} \tag{2.9}$$

It is obvious that $\zeta \in L^\infty(0, T; L^2(\Omega))$. By (2.3), we can check that

$$\int_{E_0} \langle u_n(t), \zeta(t)\rangle\mathrm{d}t = \int_0^T \langle u_n(t), \chi_{E_0}(t)\zeta(t)\rangle\mathrm{d}t \to \int_0^T \langle \chi_{E_0}(t)v^*(t), \zeta(t)\rangle\mathrm{d}t. \tag{2.10}$$

Since $\|u_n(t)\| \leq M(t)$ a.e. $t \in (0, T)$, it follows from (2.9) and (2.10) that

$$\int_{E_0} \|v^*(t)\|\mathrm{d}t = \lim_{n\to\infty} \int_{E_0} \langle u_n(t), \zeta(t)\rangle\mathrm{d}t \leq \int_{E_0} M(t)\mathrm{d}t,$$

which contradicts (2.8).

Finally, we set $u^* \triangleq \chi_{(\tau,T)}v^*$. Then (i) of this lemma follows from (2.5) and (2.6).

(ii) By (1.3), (1.4) and (i) of this lemma, we can easily check that $\tau_0 \in [0, T)$.