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Stability analysis of the telegrapher's equations with dynamic boundary condition

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ABSTRACT

This paper is concerned with the stability analysis of a distributed parameter circuit with dynamic boundary condition. The distributed parameter circuit is written by the telegrapher's equations whose boundary condition is described by an ordinary differential equation. First of all, it is shown that, for any physical parameters of the circuit, the system operator generates an exponentially stable C_0-semigroup on a Hilbert space. However, it is not clear whether the decay rate of the semigroup is the most precise one. In this paper, the spectral analysis is conducted for the system satisfying the distortionless condition, and it is shown that the semigroup satisfies the spectrum determined growth condition.

1. Introduction

Recently, for the buck converter with constant inductive load, a distributed parameter model was proposed by C. Huang et al. [1]. In that paper, a dynamic boundary condition was considered at one boundary, and the state–space representation was given in an infinite-dimensional space. However, it was not shown on whether or not the system operator generates a strongly continuous semigroup (C_0-semigroup, for short) and further the exponential stability. On the other hand, for a semilinear damped wave equation which contains the telegrapher's equation as a special case, the stabilization problem under boundary feedback has been investigated by Gugat [2], where it is shown that sufficiently small velocity damping brings large decay rates.

In this paper, we consider the distributed parameter circuit with dynamic boundary condition and discuss the stability of the system in the framework of infinite-dimensional space. Especially, we use the frequency domain approach. First of all, we show that the system operator generates an exponentially stable C_0-semigroup on a Hilbert space. Next, the spectral analysis for the system operator is conducted for the case satisfying the distortionless condition, and it is shown that the C_0-semigroup satisfies the spectrum determined growth condition by using F.L. Huang's result [3]. Here, we note that the similar method was applied by Xu et al. [4] and Kunimatsu and Sano [5] to the proof of exponential stability of the counter-flow heat exchanger equation with zero boundary conditions and the stability analysis of the counter-flow heat exchanger equation with boundary feedbacks, respectively.

The result by F.L. Huang on the spectrum determined growth condition is summarized as follows:

Theorem 1.1 ([3, Theorem 1 and Theorem 4]). *Let e^{tA} be a C_0-semigroup with the infinitesimal generator A in a Hilbert space H and set*

$$\omega_0(A) := \lim_{t\to\infty} \frac{\log \|e^{tA}\|_{\mathcal{L}(H)}}{t}, \quad \sigma_0(A) := \sup\{Re(\lambda);\ \lambda \in \sigma(A)\},$$

where $\sigma(A)$ denotes the spectrum of A. Then, the spectrum determined growth condition $\sigma_0(A) = \omega_0(A)$ is satisfied if and only if

$$\sup\{\|(\lambda I - A)^{-1}\|_{\mathcal{L}(H)};\ Re(\lambda) \geq \sigma\} < \infty$$

holds for each $\sigma > \sigma_0(A)$. Furthermore, the computational formula relating to $\omega_0(A)$ is the following:

$$\omega_0(A) = \inf\{\sigma;\ \sigma > \sigma_0(A) \text{ and } \sup\{\|(\lambda I - A)^{-1}\|_{\mathcal{L}(H)};\ Re(\lambda) \geq \sigma\} < \infty\}.$$

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2. Generation of semigroup

We shall consider the following distributed parameter circuit with dynamic boundary condition:

$$\begin{cases} L'\dfrac{\partial i}{\partial t}(t,z) = -\dfrac{\partial u}{\partial z}(t,z) - R'i(t,z), \quad t>0,\ z\in[0,l], \\ C'\dfrac{\partial u}{\partial t}(t,z) = -\dfrac{\partial i}{\partial z}(t,z) - G'u(t,z), \quad t>0,\ z\in[0,l], \\ u(t,0)=0, \quad i(t,l)=y(t), \quad t>0, \\ i(0,z)=i_0(z), \quad u(0,z)=u_0(z), \quad z\in[0,l], \\ \dot{y}(t) = -\dfrac{1}{L_l}(Ry(t)-u(t,l)), \quad t>0, \quad y(0)=y_0, \end{cases} \tag{1}$$

where $i(t,z), u(t,z)\in\mathbf{R}$ are the current and voltage at the time t and at the point $z\in[0,l]$, respectively. L_l, L' are the inductances, C' the capacitance, R, R' the resistances, G' the conductance. As is well-known, the partial differential equations are called the *telegrapher's equations*. In [1], the boundary condition $u(t,0)=U_e d(t)$ was used instead of $u(t,0)=0$, where $d(t)\in\{0,1\}$ was the switch position. In this paper, we discuss the stability of the same system under zero boundary condition, i.e., $d(t)\equiv 0$.

The system (1) is written as

$$\begin{cases} \dfrac{\partial}{\partial t}\begin{bmatrix} i(t,z) \\ u(t,z) \\ y(t)\end{bmatrix} = \begin{bmatrix} -\dfrac{R'}{L'} & -\dfrac{1}{L'}\dfrac{\partial}{\partial z} & 0 \\ -\dfrac{1}{C'}\dfrac{\partial}{\partial z} & -\dfrac{G'}{C'} & 0 \\ 0 & 0 & -\dfrac{R}{L_l}\end{bmatrix}\begin{bmatrix} i(t,z) \\ u(t,z) \\ y(t)\end{bmatrix} \\ \qquad + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{L_l}u(t,l)\end{bmatrix}, \quad \begin{bmatrix} i(0,z) \\ u(0,z) \\ y(0)\end{bmatrix} = \begin{bmatrix} i_0(z) \\ u_0(z) \\ y_0\end{bmatrix}, \\ u(t,0)=0, \quad i(t,l)=y(t). \end{cases} \tag{2}$$

Let us formulate the system (2) in a Hilbert space $X := [L^2(0,l)]^2\times\mathbf{C}$ with inner product defined by $\langle f,g\rangle_X := \langle f_1,g_1\rangle + \langle f_2,g_2\rangle + \beta f_3\overline{g_3}$ for $f=[f_1,f_2,f_3]^T\in X$, $g=[g_1,g_2,g_3]^T\in X$, where $\beta := \frac{L_l}{L'C'}$ (>0) and $\langle\varphi,\psi\rangle := \int_0^l \varphi(z)\overline{\psi(z)}dz$ for $\varphi,\psi\in L^2(0,l)$. Define the linear operator $A: D(A)\subset X\to X$ as

$$Af = \begin{bmatrix} -\dfrac{R'}{L'} & -\dfrac{1}{L'}\dfrac{d}{dz} & 0 \\ -\dfrac{1}{C'}\dfrac{d}{dz} & -\dfrac{G'}{C'} & 0 \\ 0 & 0 & -\dfrac{R}{L_l}\end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \\ f_3\end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{L_l}f_2(l)\end{bmatrix},$$

$$\begin{aligned} D(A) = \ & \{f=[f_1,f_2,f_3]^T\in[H^1(0,l)]^2\times\mathbf{C}; \\ & f_2(0)=0,\ f_1(l)=f_3\}. \end{aligned} \tag{3}$$

Then, the system (2) can be written as

$$\frac{d}{dt}\begin{bmatrix} i(t,\cdot) \\ u(t,\cdot) \\ y(t)\end{bmatrix} = A\begin{bmatrix} i(t,\cdot) \\ u(t,\cdot) \\ y(t)\end{bmatrix}, \quad \begin{bmatrix} i(0,\cdot) \\ u(0,\cdot) \\ y(0)\end{bmatrix} = \begin{bmatrix} i_0 \\ u_0 \\ y_0\end{bmatrix}, \tag{4}$$

where we assume that $i_0, u_0\in L^2(0,l)$ and $y_0\in\mathbf{R}$. Here, if the operator A generates a C_0-semigroup e^{tA} on X, then a solution of (4) is expressed as $[i(t,\cdot),u(t,\cdot),y(t)]^T = e^{tA}[i_0,u_0,y_0]^T$.

Remark 2.1. The boundary condition at $z=l$ of the system (1) can be also expressed by the form of Dirichlet integral feedback: $i(t,l) = y_0 - \frac{1}{L_l}\int_0^t(Ri(\tau,l)-u(\tau,l))d\tau$. The stabilization problem of the wave equation under Dirichlet integral feedback has been studied by Gugat [6].

First of all, we prove that the operator A generates an exponentially stable C_0-semigroup.

Theorem 2.1. *The operator A defined by (3) generates an exponentially stable C_0-semigroup e^{tA} on X.*

Proof. First, we define the operator $T\in\mathcal{L}(X)$ as $T := \begin{bmatrix} 0 & \sqrt{C'} & 0 \\ \sqrt{L'} & 0 & 0 \\ 0 & 0 & 1\end{bmatrix}$. Then, the operator $T^{-1}AT$ is expressed as

$$T^{-1}ATf = \begin{bmatrix} -\dfrac{G'}{C'} & -\dfrac{1}{\sqrt{L'C'}}\dfrac{d}{dz} & 0 \\ -\dfrac{1}{\sqrt{L'C'}}\dfrac{d}{dz} & -\dfrac{R'}{L'} & 0 \\ 0 & 0 & -\dfrac{R}{L_l}\end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \\ f_3\end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{\sqrt{L'}}{L_l}f_1(l)\end{bmatrix},$$

$$\begin{aligned} D(T^{-1}AT) = \{ & f=[f_1,f_2,f_3]^T\in[H^1(0,l)]^2\times\mathbf{C}; \\ & f_1(0)=0,\ f_2(l)=\frac{1}{\sqrt{C'}}f_3\}. \end{aligned}$$

Here, note that the matrix on the right-hand side of $T^{-1}ATf$ is symmetric. Then, since it follows that $\mathrm{Re}\langle T^{-1}ATf,f\rangle_X = -\frac{G'}{C'}\|f_1\|^2 - \frac{R'}{L'}\|f_2\|^2 - \beta\frac{R}{L_l}|f_3|^2$ for all $f=[f_1,f_2,f_3]^T\in D(T^{-1}AT)$, we see that

$$\mathrm{Re}\langle T^{-1}ATf,f\rangle_X \le -\gamma\|f\|_X^2, \quad \forall f\in D(T^{-1}AT), \tag{5}$$

where $\gamma := \min\{\frac{R'}{L'},\frac{G'}{C'},\frac{R}{L_l}\}$ (>0). Also, as shown in Appendix, there exists a number $\lambda>0$ such that the range of $\lambda I - T^{-1}AT$ is equal to X, i.e.,

$$R(\lambda I - T^{-1}AT) = X. \tag{6}$$

Since it is easily verified that $T^{-1}AT$ is a densely defined closed linear operator, it follows from the fact together with (5) and (6) that the operator $T^{-1}AT+\gamma I$ generates a contractive C_0-semigroup $e^{t(T^{-1}AT+\gamma I)}$ on X, i.e., the operator $T^{-1}AT$ generates an exponentially stable C_0-semigroup $e^{t(T^{-1}AT)}$ with norm bound $\|e^{t(T^{-1}AT)}\|_{\mathcal{L}(X)}\le e^{-\gamma t}$ on X, by using the Lumer–Phillips' Theorem [7, Theorem 1.4.3] (see also the proof of [8, Corollary 2.2.3]). That is, the operator A generates an exponentially stable C_0-semigroup e^{tA} with the same decay rate on X. □

Remark 2.2. By using the inner product with $\beta=1$ instead of $\beta=\frac{L_l}{L'C'}$, we can also show that the operator A generates a C_0-semigroup on X by applying the perturbation result of semigroups [9, Corollary III.1.5], but, in such a way, we cannot conclude the exponential stability of the C_0-semigroup.

3. Stability analysis

In this section, we show that, under some assumption with respect to the physical parameters of the circuit, one can give the most precise decay rate for the C_0-semigroup e^{tA} shown in Theorem 2.1, by analyzing the spectrum of A as well as using F.L. Huang's result (Theorem 1.1).

Assumption 3.1. The condition $R'C' = L'G'$ is satisfied. Hereafter, set $\omega := \frac{G'}{C'} = \frac{R'}{L'}$ (>0).