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Review

Tractable sufficient stability conditions for a system coupling linear transport and differential equations ☆

Mohammed Safi *, Lucie Baudouin, Alexandre Seuret

LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France

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ABSTRACT

This paper deals with the stability analysis of a system of finite dimension coupled to a vectorial transport equation. We develop here a new method to study the stability of such a system, coupling ordinary and partial differential equations, using linear matrix inequalities led by the choice of an appropriate Lyapunov functional. To this end, we exploit Legendre polynomials and their properties, and use a Bessel inequality to measure the contribution of our approximation. The exponential stability of a wide class of delay systems is a direct consequence of this study, but above all, we are detailing here a new approach in the consideration of systems coupling infinite and finite dimensional dynamics. The coupling with a vectorial transport equation is a first step that already prove the interest of the method, bringing hierarchized conditions for stability. We will give exponential stability results and their proofs. Our approach will finally be tested on several academic examples.

Contents

1. Introduction

Systems coupling partial and ordinary differential equations are one type of infinite dimensional systems. The robust control of what is also called distributed parameter systems has been a very active field for the last decades and has spawned several branches, such as e.g. in stability analysis and stabilization design. This article is meant to perform a stability analysis of a system of linear ordinary differential equations (ODE) coupled to a vectorial transport equation, which is a first order hyperbolic partial differential equation (PDE). Analysing and controlling this type of system coupling ODE and PDE is an attractive topic at the interface of applied mathematics and automatic control. A large number of papers already exist on stability of this class of systems: see e.g. [1–5] among many

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* Corresponding author.
E-mail address: msafi@laas.fr (M. Safi).

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others. For instance, such systems appear in the context of energy management as in [6]. The coupled system we will study in this article has the specificity to represent an alternative formulation of a time-delay system (TDS) with a vectorial transport equation replacing the delay terms. One should know that TDS also have a solution that evolves in an infinite dimensional space and in our opinion, this is an interesting connection that suggests to benefit of the different approaches for the stability and control in both domains (TDS and PDE). On the one hand, we can refer to several stability and control studies of PDE such as in the books [7–9], and the references therein, or the non-exhaustive list of articles [10–12] and [13]. On the other hand, TDS have been widely investigated in the literature (see e.g. [14–18], and [19]), and used in many areas, as in biological systems, mechanical transmissions or networked control systems.

One of the most fruitful fields of research in stability of these TDS relies on the exhibition of Lyapunov–Krasovskii functionals (LKF). In reference [16], the candidate Lyapunov functional called *complete LKF*, leads even to a necessary and sufficient stability condition. Nevertheless, the parameters composing this complete LKF make it numerically difficult to handle, especially for high dimensional systems [20,21]. A lot of investigations then turns to approximating these parameters, and more recently, approximation methods have been improved by considering polynomial like parameters of arbitrary degree [22].

Our goal in this article is to provide a novel framework to address stability problem of linear coupled finite/infinite dimensional systems following recent advances on time-delay systems presented for instance in [18]. Our work here is a first step towards more general PDE and focusses on the transport equation case, giving rise to a necessary stability criteria of stability when such a PDE is coupled to an ODE. The analysis adopted in the present paper is based on the Lyapunov theorem for infinite dimensional systems, which, according to us, represents a first relevant challenge. As a consequence of this more general analysis, we will provide a unified set of linear matrix inequalities (LMI) conditions allowed to guarantee exponential stability (in the sense of the L^2-norm) applicable to a wide class of delay systems including single/multiple/cross-talking delays for differential and difference equations, as particular case but is not only resume to these classes of systems. The objective of this paper is to provide a new framework for the analysis of this linear coupled ODE/hyperbolic PDE system. This contribution extends our preliminary studies presented in [23] and [24], where only a single transport speed was considered. The main difficulty is related to the infinite dimensional state of the system, which prevents from extending directly the existing methods of the finite dimension analysis. Nevertheless, in order to provide efficient and tractable stability conditions, we employ a polynomial approximation of the state expressed using Legendre polynomials, following the approach developed for TDS in [18].

Notations: $\mathbb{N}$ is the set of positive integer, $\mathbb{R}^n$ is the n-dimensional Euclidean space with vector norm $|\cdot|_n$. I_n is the identity matrix in $\mathbb{R}^{n\times n}$, $0_{n,m}$ the null matrix $\in \mathbb{R}^{n\times m}$, $\begin{bmatrix} A & B \\ \star & C \end{bmatrix}$ replaces the symmetric matrix $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$. We denote $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$ (resp. $\mathbb{S}^n_+ = \{P \in \mathbb{S}^n, P \succ 0\}$, and $\mathbb{D}^n_+$) the set of symmetric (resp. symmetric positive definite and diagonal positive definite) matrices and diag(A, B) is a bloc diagonal matrix. For any square matrix A, we define $\text{He}(A) = A + A^\top$. Finally, $L^2(0, 1; \mathbb{R}^m)$ represents the space of square integrable functions over the interval $[0, 1] \subset \mathbb{R}$ with values in $\mathbb{R}^m$ and the partial derivative in time and space are denoted ∂_t and ∂_x, while the classical derivative are $\dot{X} = \frac{d}{dt}X$ and $\mathcal{L}' = \frac{d}{dx}\mathcal{L}$.

2. Formulation of the problem

2.1. Linear coupled ODE–PDE system

This article is devoted to the stability analysis of a system of ODEs coupled with a vectorial transport equation that takes the following shape:

$$\begin{cases} \dot{X}(t) = AX(t) + Bz(1, t), & t > 0, \\ \partial_t z(x, t) + \Lambda \partial_x z(x, t) = 0, & x \in (0, 1), t > 0, \\ z(0, t) = C_1 X(t) + C_2 z(1, t), & t > 0, \\ z(x, 0) = z^0(x), & x \in (0, 1) \\ X(0) = X^0. \end{cases} \tag{1}$$

The state of this coupled system is composed of $X(t) \in \mathbb{R}^n$ and $z(\cdot, t) \in L^2(0, 1; \mathbb{R}^m)$. A, B, C_1 and C_2 are constant matrices with appropriate dimensions. The matrix of propagation speed $\Lambda \in \mathbb{D}^m_+$ is given by:

$$\Lambda = \text{diag}(\rho_i I_{m_i})_{\{i=1\ldots p\}}. \tag{2}$$

Thus, each velocity $\rho_i > 0$ is applied to m_i components of the state $z(x, t)$ such that $m = \sum_{i=1}^p m_i$. Note that the situation of negative transport speed $\rho_i < 0$, for some $i \le p$ can be recast in the same formulation with positive transport speed using a change of state spatial variable $x' = 1 - x$. The transport equation $\partial_t z + \Lambda \partial_x z = 0$ in (1) of unknown $z = z(x, t)$ is a simple linear vectorial PDE and if the initial data $z^0 \in L^2(0, 1; \mathbb{R}^m)$ and the lateral boundary data $z(0, \cdot) = u \in L^2(\mathbb{R}_+; \mathbb{R}^m)$ are given, it has a unique solution $z \in C(\mathbb{R}_+; L^2(0, 1; \mathbb{R}^m))$ such that (see e.g. [9]), for all $t > 0$:

$$\|z(t)\|_{L^2(0,1;\mathbb{R}^m)} \le K(\|z^0\|_{L^2(0,1;\mathbb{R}^m)} + \|u\|_{L^2(\mathbb{R}_+;\mathbb{R}^m)}).$$

Considering now the finite dimensional system in $X(t)$ coupled to the transport equation in the variable $z(x, t)$, we can notice that the coupling is linear and the existence of solution can be proved thanks to Theorem A.6 in [7]. Following this theorem, for every $z^0 \in L^2(0, 1; \mathbb{R}^m)$ and $X^0 \in \mathbb{R}^n$, the Cauchy problem (1) has a unique solution. Moreover, there exist $K > 0$ and $\delta > 0$ such that the solution $(z(x, t), X(t))$ of system (1) satisfies :

$$\|X(t)\| + \|z(t)\|_{L^2(0,1;\mathbb{R}^m)} \le Ke^{\delta t}.$$

This well-posedness result suggests the choice of the following total energy of the system $E(X(t), z(t)) = |X(t)|^2_n + \|z(t)\|^2_{L^2(0,1;\mathbb{R}^m)}$, and in the sequel, we will denote $E(t) = E(X(t), z(t))$ in order to simplify the notations.

2.2. Lyapunov stability

We are looking for a candidate Lyapunov functional for (1) of the shape:

$$\begin{aligned} V(X(t), z(t)) = {} & \int_0^1\int_0^1 \begin{bmatrix} X(t) \\ z(x_1, t) \end{bmatrix}^\top \begin{bmatrix} P & \mathcal{Q}(x_1) \\ \mathcal{Q}^\top(x_2) & \mathcal{T}(x_1, x_2) \end{bmatrix} \\ & \times \begin{bmatrix} X(t) \\ z(x_2, t) \end{bmatrix} dx_1 dx_2 \\ & + \int_0^1 z^\top(x, t) e^{-2\delta x \Lambda^{-1}} (S + (1 - x)R) z(x, t) dx, \end{aligned} \tag{3}$$

where the scalar $\delta > 0$, the matrices $P \in \mathbb{S}^n_+$, $S, R \in \mathbb{S}^m_+$ and the functions $\mathcal{Q} \in L^2(0, 1; \mathbb{R}^{n\times m})$ and $\mathcal{T} \in L^\infty((0, 1)^2; \mathbb{S}^m)$ have to be determined. As for the energy, in the sequel, we will denote $V_N(t) = V_N(X(t), z(t))$ in order to simplify the notations.

Remark 1. Since the transport speed matrix belongs to $\mathbb{D}^m_+$, it is invertible and admits an inverse matrix Λ^{-1} given by $\Lambda^{-1} = \text{diag}(\rho_i^{-1} I_{m_i})_{\{i=1\ldots p\}}$. We can define its exponential matrix $e^{\Lambda^{-1}} =$