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Hamiltonian systems discrete-time approximation: Losslessness, passivity and composability[☆]

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ABSTRACT

In this paper a *passive integrator* dedicated to input/output Hamiltonian systems approximation is presented. In a first step, a discrete Hamiltonian framework endowed with a Lie derivative-like formula is introduced. It is shown that the discrete dynamics encodes energy conservation and passivity. Additionally, the characterization of the discrete dynamics in terms of Dirac structure is shown to be invariant by interconnection. The class is thus composable: networked systems belong to the class. In a second step, the discrete dynamics is considered as a one-step integration method. The method is shown to be convergent and provides a discrete-time approximation of an input/output Hamiltonian system. Accordingly, the discrete dynamics inherits intrinsic energetic characteristics (storage function and dissipation rate) from the original system. The method is thus tagged as passive integrator. As an illustration, the closed-loop behavior of interconnected subsystems and the stabilization of a rigid body spinning around its center of mass are presented.

1. Introduction

It is well-known that Lagrangian and Hamiltonian dynamics deserve dedicated time integration methods driven by the preservation of invariants. Mostly, one is concerned either with structure-preserving or with Hamiltonian-preserving algorithms. Accordingly, approximation schemes fall into one of the two categories: geometric or energetic integrators.

Such integrators have been raised by numerical simulation issues such as long time simulation, stability and efficiency of integration methods regarding system's invariants. There, the authors develop and refine energetic [1–5], geometric [6–8] or multi-symplectic [9,10] integrators. Among the extensive literature available with many points of view, only few references are given here. Standard textbooks are [11–13]. Notice this active area of research focuses on **unforced** Hamiltonian systems, *i.e.* without input/output.

From an automatic control viewpoint, systems are endowed with input/output variables that allow to describe interactions with the surrounding environment. These interactions thus need to be soundly encoded by the time-stepping algorithm. Restricted to this field, the literature is much less extensive. Note the above mentioned integrators are not able to handle input/output variables. This paper precisely addresses this problem regarding **passive** Hamiltonian systems.

As closed-loop stability is concerned with passivity, stabilizing control law synthesis relies on a passivity equation. However, general discrete-time stabilization results [14,15] are derived from given discrete-time dynamics *assumed to be passive*. The approximation procedure is not considered. Restricted to input/output Hamiltonian systems, the discrete-time approximation methods proposed in the literature only achieve a **truncated energy balance equation** (*e.g.* discrete-time model and control [16,17], sampled-data [18,19]).

An alternative to a direct discretization of the differential equations resides on the discrete translation of the geometry of modeling tools. The resulting discrete geometry framework has shown relevant insights [20–24]. Again, only unforced systems were considered. Relative to input/output Hamiltonian systems, a similar idea has been used to deal with the Dirac description of the equations [25–28]. Although additional structural properties were captured, the time discretization step has not been treated to obtain a class of discrete systems with intrinsic properties of passive Hamiltonian systems. Consequently, missing passivity and composability loses modeling and control technics within port-Hamiltonian framework.

To overcome this drawback, we propose a discrete Hamiltonian framework defined by a class of discrete systems shown to be

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passive and composable. It turns out that this discrete framework is naturally endowed with a Lie derivative-like formula. The discrete dynamics is obtained by a translation of balance equations thanks to discrete gradient introduced in [2]. Known as an energetic integrator dedicated to unforced systems, the discrete gradient is here used to define a **passive integrator** to cope with input/output Hamiltonian systems. The resulting discretization scheme is shown to be a convergent one-step method which provides a discrete-time dynamics with inherited energetic characteristics (storage function and dissipation rate).

The paper is organized as follows. Section 2 motivates a proper properties encoding in terms of closed-loop convergence rate and networked systems modeling. Our contributions are presented in Sections 3 and 4. The discrete Hamiltonian framework is defined in Section 3. The passivity equation and the Dirac structure characterization are given. The passive integrator is studied in Section 4 where its convergence is analyzed. Section 5 is concerned with numerical simulations.

2. Motivations

This paper is concerned with the discrete-time approximation problem of the following class of passive systems (stemming from port-Hamiltonian systems [29] satisfying integrability conditions).

Definition 1 (*[30]*). Given H a $\mathcal{C}^1$ real-valued function on $\mathbb{R}^m$ assumed to be bounded from below, a *passive Hamiltonian system*, denoted by Σ_H, is defined by the following equations:

$$\Sigma_H\ :\ \begin{bmatrix}\dot{x}\\ -y\end{bmatrix}=\begin{bmatrix}[J-R](x) & g(x)\\ -g^T(x) & 0\end{bmatrix}\begin{bmatrix}\nabla H(x)\\ u\end{bmatrix}, \tag{1}$$

where $J=-J^T:\mathbb{R}^m\to\mathbb{R}^{m\times m}$ is called the *structure* matrix, $R=R^T:\mathbb{R}^m\to\mathbb{R}^{m\times m}$ with $R\geq 0$ is the *dissipation* matrix, $g:\mathbb{R}^m\to\mathbb{R}^{m\times l}$ the *input* matrix and (u,y) the *passive inputs/outputs*.

Passive Hamiltonian systems are associated with an energy-based approach of physical systems where a system is considered as a network of interconnected subsystems. Each subsystem describes the energy exchanges relative to elementary phenomena interacting with its surrounding (the network nodes and bonds). Interactions in a whole are described as a set of constraints given by the interconnection structure (the network topology).

Systems (1) are known to define a class of dynamical systems with the following inherent properties:

(P1) Every element in the class verifies a passive equation with an explicit dissipation rate.
(P2) The class is invariant by interconnection.

We shall seek to clone these properties at a discrete level. Let us first recall their fundamental interest from an energy-based viewpoint.

Regarding property (P1), the derivative of H along the trajectories of (1) gives a passivity equation with storage function H:

$$\frac{\mathrm{d}}{\mathrm{d}t}H(x)=y^Tu-\|\nabla H(x)\|_R^2\quad\leq y^Tu\,, \tag{2}$$

for all input function u, where $\|\cdot\|_R$ stands for the (semi-)norm associated with the (semi-)positive definite matrix R. The term $\|\nabla H(x)\|_R^2$ thus reads as the *dissipation rate* of the system. Observe the passivity equation (2) also encodes losslessness when $R\equiv 0$, and further, energy conservation in the unforced case ($R\equiv 0$ and $u\equiv 0$ imply $\frac{\mathrm{d}}{\mathrm{d}t}H=0$).

Since the dissipation rate is related to the convergence rate when applying passivity-based control design technics, if the amount of dissipated energy is numerically badly captured, a numerical energy drift occurs and the resulting closed-loop behavior is incorrectly estimated. Consequently, spurious simulation results such as limit cycle and unstable trajectory can be obtained as illustrated in [31].

Regarding property (P2), it can be shown that a network of passive Hamiltonian systems $\left\{\Sigma_{H_i}\right\}_i$ constructed by constraining the pairs $\{(u_i,y_i)\}_i$ through a (constant) Dirac structure, is a (generalized) Dirac structure having (1) as coordinate representation (see [32]). We shall use the terminology *composable* to refer this invariance property of the class.

From a discrete viewpoint, mimicking composability would mean that both the discretized network of interconnected subsystems and the network of discretized subsystems produce identical dynamics. On the contrary, the discrete dynamics depends on the partitioning of the network system. One may therefore vainly wonder about the relevance of the choice of the partition regarding the discrete dynamics obtained. An illustration is given for the negative feedback interconnection of linear Hamiltonian systems in [33].

In the sequel, a class of discrete Hamiltonian systems that verifies properties (P1) and (P2) is introduced. Next, it is shown that these systems are a discrete-time approximation of passive Hamiltonian systems (1). So we claim that the process as a whole depicts a relevant discrete framework for the class considered.

3. Discrete Hamiltonian framework

In this section, it is defined a class of discrete Hamiltonian systems which is shown to satisfy properties (P1) and (P2).

The underlying idea is to translate the energy exchanges described in (1) by replacing the gradient ∇ by the discrete gradient $\overline{\nabla}$ defined in what follows.

Definition 2 (*[2]*). Let $f=(f_1,\ldots,f_n)$ be a $\mathcal{C}^1$-map from $\mathbb{R}^m$ to $\mathbb{R}^n$. The *difference quotient* of f, denoted by $\overline{\nabla}f$, is a matrix-valued map from $\mathbb{R}^m\times\mathbb{R}^m$ to $\mathbb{R}^{n\times m}$ defined by

$$\begin{aligned}\overline{\nabla}_jf_i(x,x')=\frac{1}{x'_j-x_j}\big[&f_i(x'_1,\ldots,x'_{j-1},\ x'_j\ ,x_{j+1},\ldots,x_m)\\ &-f_i(x'_1,\ldots,x'_{j-1},\ x_j\ ,x_{j+1},\ldots,x_m)\big]\end{aligned} \tag{3}$$

where $1\leq i\leq n$ and $1\leq j\leq m$.

Note $\overline{\nabla}f(x,x')$ is well-defined as $\|x'-x\|\ \to\ 0$ since f is $\mathcal{C}^1$. Moreover, the following identity is satisfied:

$$\overline{\nabla}f(x,x')(x'-x)=f(x')-f(x)\,, \tag{4}$$

for all x,x' in $\mathbb{R}^m$ expressed as column vectors.

Let us now introduce discrete notations. Set $X:\mathbb{N}\to\mathbb{R}^m$ the discrete state column vector. That is, for all k in $\mathbb{N}$, $X(k)$ is the m-dimensional state vector at stage k given by $X(k)=\begin{bmatrix}X_1(k) & \cdots & X_m(k)\end{bmatrix}^T$. A discrete trajectory is then given by a matrix $X^N\in(\mathbb{R}^m)^N$ where N is the number of iterates. Denote by $X':\mathbb{N}\to\mathbb{R}^m$ the map $k\mapsto X'(k)=X(k+1)$. Given a real $\Delta t>0$, one defines the *state rate variation* as $\overline{\nabla}X=\frac{X'-X}{\Delta t}$.

Further, given H a $\mathcal{C}^1$ real-valued function on $\mathbb{R}^m$, one defines its *discrete gradient* along X^N as $\overline{\nabla}H(X,X')$. Its column vector expression is denoted by $\overline{\nabla}^TH$.

We introduce the following class of discrete systems.

Definition 3. With the above notations, a *discrete passive Hamiltonian system*, denoted by $\overline{\Sigma}_H$, is defined by the equations

$$\overline{\Sigma}_H\ :\ \begin{bmatrix}\overline{\nabla}X\\ -y\end{bmatrix}=\begin{bmatrix}[J-R](X) & g(X)\\ -g^T(X) & 0\end{bmatrix}\begin{bmatrix}\overline{\nabla}^TH(X,X')\\ u\end{bmatrix} \tag{5}$$

with $J=-J^T:\mathbb{R}^m\to\mathbb{R}^{m\times m}$, $R=R^T:\mathbb{R}^m\to\mathbb{R}^{m\times m}$ satisfies $R\geq 0$ and $g:\mathbb{R}^m\to\mathbb{R}^{m\times l}$.