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Imaginary axis eigenvalues of matrix delay equations with a certain alternating coefficient structure

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ABSTRACT

In this paper we examine the imaginary axis eigenvalues of matrix delay differential equations with coefficients alternating between Hermitian and skew Hermitian, such as occurring in many mechanical systems. We show that given this special structure and a certain sign condition, the dimension of the eigenvalue problem is greatly reduced. We also give some theorems on eigenvalue crossing directions in this context, and show how reduction of dimension and crossing theorems work to provide insight on mechanical systems with time delay.

1. Introduction

In this paper we will be interested in determining the imaginary axis eigenvalues of delay differential equations with a certain algebraic structure. Our particular topic will be matrix high order differential equations with a single delay and with coefficients alternating between Hermitian and skew Hermitian. The reason for this choice of coefficient structure will become apparent from fundamental observations involving the generalized eigenvalues of a matrix pair having one matrix Hermitian and the other sign semi-definite. We will see that with this structure the dimension of the pure imaginary eigenvalue problem is often markedly reduced, and it will be possible to examine certain related questions with insight rather than computation alone.

The choice of single delay will keep us from too much engagement in subtleties relating exponential variables to the various properties of linear functional differential equations. In this first presentation on the subject, familiarity has been a consideration in selecting examples. We will use matrix delay equations which are second order in the derivative and have well-known interpretations in mathematical sciences.

The idea that the oscillation frequencies of linear autonomous time-delay equations are a key to their stability properties is well-established, and there are several known approaches to determining these frequencies in matrix single delay equations [1–5]. Noting its fundamental nature, the topic continues to appear in publications, for example in bifurcation numerics [6], in integral delay and distributed delay equations [7,8], or in considering the part played by operator averaging in delay systems with rapid periodic time-dependence [9], and generally in the modern theory of delay differential equations [10]. Practical interest in the topic is evident in books on time-delay control systems [11,12]. Investigations making use of special structure are less frequent, unless we count practical modeling studies among these, e.g. those occurring in vibration suppression [13]. Although we hope that this contribution is used as a starting point for mathematical scientists with both theoretical and applied interests, we will, as noted, emphasize familiar matrix delay equations when giving examples.

In the next section we define our items of interest and give the fundamental lemmas that make transparent the subsequent theorems on dimensionality and the question of oscillation eigenvalues. These are given in Section 3, where we see that certain matrix delay equations with our kind of alternating coefficient structure have only unity or the opposite for associated exponentials. The next section looks at the direction of eigenvalue crossing, an important topic for examples. Section 5 is given over to examples and some discussion. Conclusions are given in Section 6.

2. Matrix delay equations and matrix polynomials

Here we give the basic definitions and observations from linear algebra which pave the way for the theorems of the next section. We define what we mean by a matrix polynomial with alternating coefficient structure, and give some routine facts relating such matrix polynomials to the type of matrix delay differential equation we investigate with them. We briefly examine the generalized eigenvalue relations holding between semi-definite or sign definite matrices and Hermitian matrices, supplementing the well-known fact that the product of a sign definite matrix and a Hermitian matrix has real eigenvalues.

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Before we proceed, it will be useful to have the notation $\mathbb{C}_-$, $i\mathbb{R}$, $\mathbb{C}_+$ for the standard partitioning of the complex plane. Also, we shall use $\lambda_{max}(\cdot)$ and $\lambda_{min}(\cdot)$, respectively, to designate the maximum and minimum eigenvalues of a Hermitian matrix. Finally, d_t will denote the time differentiation operator, and d_s will denote the differentiation operator with respect to the complex variable s.

Consider matrix delay differential equations of the form

$$\sum_{k=0}^{p} J_k x^{(k)}(t) = \sum_{k=0}^{r} E_k x^{(k)}(t-h) \tag{1}$$

where each $J_k, E_k \in \mathbb{C}^{n\times n}$, $p \geq 1$, $r \geq 0$ and $J_p \neq 0$, $E_r \neq 0$. In practice readers will usually be interested in cases where J_k, $E_k \in \mathbb{R}^{n\times n}$, and our examples will emphasize this. For complex s, z, we will find it convenient to write

$$J(s) = \sum_{k=0}^{p} J_k s^k,\ E(s) = \sum_{k=0}^{r} E_k s^k \tag{2}$$

$$T(s,z) = J(s) - zE(s). \tag{3}$$

Note that with this notation, (1) can be written as $J(d_t)x(t) = E(d_t)x(t-h)$.

If $x(t) = e^{st}v$ is a solution of (1), $v \in \mathbb{C}^n - \{0\}$, we refer to $x(t)$ as a spectral solution. We then have $T(s, e^{-hs})v = 0$. We say that s is an eigenvalue of (1), and that $T(s, e^{-hs})$, $detT(s, e^{-hs})$ are the **characteristic matrix function** and **scalar characteristic function** of (1), respectively. We will be interested in determining those eigenvalues s of (1) which lie on the imaginary axis, i.e. in finding pure imaginary solutions of $|T(s, e^{-hs})| = 0$. For such s, we know that $z = e^{-hs}$ lies on the complex unit circle, i.e. in $\{z \in \mathbb{C} : |z| = 1\}$.

Definition 2.1. A matrix polynomial $A(s) = \sum_{k=0}^{m} A_k s^k$ is said to have **alternating coefficient structure** if one of the following two conditions is true:
1. $A_k^* = A_k$ if k is even, $A_k^* = -A_k$ if k is odd **2**. $A_k^* = -A_k$ if k is even, $A_k^* = A_k$ if k is odd.
In the first case we say that $A(s)$ has alternating coefficient structure with **Hermitian** constant term, while in the second we say that $A(s)$ has alternating coefficient structure with **skew Hermitian** constant term.

Note that the following are equivalent:
1. $A(s)$ has alternating coefficient structure with Hermitian constant term;
2. $iA(s)$ has alternating coefficient structure with skew Hermitian constant term ($i = \sqrt{-1}$).

Definition 2.2. A matrix delay differential equation of the form (1) is said to satisfy **Hypothesis H** if both of the matrix polynomials $J(s)$ and $E(s)$ have alternating coefficient structure with Hermitian constant term.

It is worth noting immediately that if both $J(s)$ and $E(s)$ have alternating coefficient structure with skew Hermitian constant term, then multiplication of both sides of (1) by i gives us a matrix delay differential equation which satisfies **Hypothesis H**. This is convenient for examples and applications, since it allows for conversion to a common form.

Lemma 2.1. *Consider $A(s) = \sum_{k=0}^{m} A_k s^k$, each $A_k \in \mathbb{C}^{n\times n}$. Conditions **1**, **2** below are equivalent, and likewise equivalent are conditions **3**, **4**.*

***1**. $A(s)$ has alternating coefficient structure with Hermitian constant term.*

***2**. $A(s)^* = A(s)$ for all imaginary axis s.*

***3**. $A(s)$ has alternating coefficient structure with skew Hermitian constant term.*

***4**. $A(s)^* = -A(s)$ for all imaginary axis s.*

Proof. **1** ⇒ **2**: Suppose we have **1**, and let $s \in i\mathbb{R}$. If k is even, then s^k is real, thus $A_k s^k$ is Hermitian. If k is odd, then s^k is pure imaginary, thus $A_k s^k$ is again Hermitian. We conclude that $A(s)^* = A(s)$.
2 ⇒ **1**: Suppose $A(s)^* = A(s)$ throughout $i\mathbb{R}$. Then $\sum_{k=0}^{m} A_k^*(-s)^k = \sum_{k=0}^{m} A_k s^k$ throughout $i\mathbb{R}$. Matching polynomial coefficients, we see that $A_k^* = A_k$ for even k and $A_k^* = -A_k$ for odd k.
3 ⇔ **4**: The proof is similar to the proof of "**1** ⇔ **2**". □

We will have use for a matrix pair which is companion to $A(s)$.

Definition 2.3. Given a matrix polynomial $A(s) = \sum_{k=0}^{m} A_k s^k$, we define the **companion pair** of A, or ***comp***(A), as follows:
1. If $A(s)$ is degree 1 in s, then ***comp***(A) is the matrix pair $(-A_0, A_1)$.
2. If $A(s)$ is degree 2 or higher in s, then ***comp***$(A) = (-C_0, C_1)$, with

$$C_1 = \begin{bmatrix} A_m & 0 \\ 0 & I_{(m-1)n} \end{bmatrix},\quad C_0 = \begin{bmatrix} A_{m-1} \dots & A_0 \\ -I_{(m-1)n} & 0 \end{bmatrix}. \tag{4}$$

Note that the generalized eigenvalues of $comp(A)$ are the complex numbers s for which $A(s)$ is singular.

The next two lemmas and the corollary focus on the interplay between Hermitian matrices and matrices which are sign definite or semi-definite. With this lens the theorems of the following section will be readily apparent.

Lemma 2.2. *Let the complex matrix E be Hermitian and semi-definite, and let $x \in \mathbb{C}^n$. Then $x^*Ex = 0$ if, and only if, $Ex = 0$.*

Proof. Suppose E is semi-definite, $E \neq 0$. Note that $E = U^*DU$ with U unitary and $D = diag(D_r, 0_{n-r})$; here D_r is sign definite and real diagonal of dimension r. Setting $y = Ux$, write $x^*Ex = 0$ as $y^*Dy = 0$, and note that $y^*Dy = 0$ is equivalent to $Dy = 0$. This in turn is equivalent to $U^*Dy = 0$, i.e. to $Ex = 0$. In conclusion, $x^*Ex = 0$ is equivalent to $Ex = 0$. □

Lemma 2.3. *Let A, E be Hermitian matrices and let E be semi-definite. Then exactly one of the following holds:*

***1**. all solutions λ of $|\lambda E - A| = 0$ are real*

***2**. $Ker(E) \cap Ker(A) \neq (0)$, thus $|\lambda E - A| = 0$ for all complex λ.*

Proof. Suppose $|\lambda E - A| = 0$. We have nonzero $v \in \mathbb{C}^n$ with $\lambda Ev = Av$, so that both $v^*Av = v^*\lambda Ev = \lambda v^*Ev$ and $v^*Av = (A^*v)^*v = (Av)^*v = (\lambda Ev)^*v = \bar{\lambda}v^*E^*v = \bar{\lambda}v^*Ev$. Thus $\lambda v^*Ev = \bar{\lambda}v^*Ev$.

Now if λ is not real, then $\bar{\lambda} \neq \lambda$, and $v^*Ev = 0$. From Lemma 2.2 we then see $Ev = 0$, and $0 = \lambda Ev = Av$. We conclude that $v \in Ker(A) \cap Ker(E)$ if $|\lambda E - A| = 0$ and λ is not real, i.e. $Ker(E) \cap Ker(A) \neq (0)$ if there is a nonreal solution λ to $|\lambda E - A| = 0$. □

Corollary 2.1. *Let A, E be Hermitian matrices and let E be sign definite. Then all solutions of $|\lambda E - A| = 0$ are real.*

Proof. In this case $Ker(E) = (0)$, thus $Ker(E) \cap Ker(A) = (0)$, and the only possibility in Lemma 2.3 is "**1**". □

Note that Corollary 2.1 returns us to the fact that the product of a sign definite matrix (E^{-1}) and a Hermitian matrix (A) has real eigenvalues.

3. Main propositions

Prior to our main propositions, we open with some guidance on whether the eigenvalues proposed in our necessary conditions are true eigenvalues. Discounting the case where a proposed eigenvalue is zero but its associated exponential must be minus unity, we will see that the necessary conditions given later are also sufficient. Then we proceed to our main propositions, which are fairly evident from our preparation in Section 2.