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Converting high order linear PDEs to first order: Noncommutative case

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ABSTRACT

A straightforward way is offered to convert a system of high order linear functional equations of various type into a system of first order equations.

1. Introduction

Recently in [1], we have described a simple technique for converting high order constant coefficient linear partial differential equations to first order equations, and our goal in this note is to extend that technique to much more general classes of linear functional equations including, say, the class of linear partial differential equations with polynomial coefficients.

Following many authors, we shall use the setup of Ore algebras, which allows to treat in a uniform way all the various linear functional equations that are of interest for linear control (see Chyzak, Quadrat and Robertz [2,3], Seiler and Zerz [4]).

Throughout, A is an *arbitrary* commutative ring, $\sigma = (\sigma_1, \dots, \sigma_r)$ is a sequence of automorphisms of the ring A and $\delta = \{\delta_1, \dots, \delta_r\}$ is a sequence, where each δ_i is a derivation of A with respect to σ_i (i.e. δ_i is an additive endomorphism of A such that for $a, b \in A$, we have $\delta_i(ab) = \sigma_i(a)\delta_i(b) + \delta_i(a)b$). We require that all the maps σ_i, δ_j commute with each other:

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad \delta_i\delta_j = \delta_j\delta_i, \quad \sigma_i\delta_j = \delta_j\sigma_i$$

for $i \neq j$.

Let $\partial = (\partial_1, \dots, \partial_r)$ be a sequence of variables. The Ore algebra $D = A[\sigma, \delta; \partial]$ is the algebra over A generated by $\partial_1, \dots, \partial_r$ subject to the following relations

$$\partial_k a = \sigma_k(a)\partial_k + \delta_k(a) \quad (a \in A,\ 1 \leq k \leq r),$$

$$\partial_i\partial_j = \partial_j\partial_i \quad (1 \leq i, j \leq r).$$

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This is a noncommutative ring in general. Noncommutativity takes place between generators and base ring elements only; different generators commute with each other. Elements of D are called Ore polynomials.

A typical example of Ore algebra is the so-called rth Weyl algebra

$$\mathbb{C}[t_1, \dots, t_r]\langle\partial_1, \dots, \partial_r\rangle$$

with $\sigma_i = id$ and $\delta_i = d/dt_i$ for all i.

Recall that a multi-index is an r-tuple of nonnegative integers, i.e., an element of $\mathbb{Z}_+^r$. If $i = (i_1, \dots, i_r)$ is a multi-index, write ∂^i for $\partial_1^{i_1} \dots \partial_n^{i_r}$. If n is a nonnegative integer, let $\Delta(n)$ denote the set of multi-indices of order less than or equal to n. (The order of $i = (i_1, \dots, i_r)$ is defined to be $|i| = i_1 + \dots + i_r$.)

Due to the defining relations, any Ore polynomial can be written (uniquely) in the left normal form, i.e., as a "left" polynomial $\sum_i a_i\partial^i$. It is easily seen that

$$a\partial_k = \partial_k\sigma_k^{-1}(a) - \delta_k(\sigma_k^{-1}(a)), \tag{1}$$

and consequently any Ore polynomial can be written also in the right normal form, i.e., as a "right" polynomial $\sum_i \partial^i a_i$.

Using the normal forms, one defines in an obvious way the notion of degree for Ore polynomials. (It does not depend on whether an Ore polynomial is represented as a "left" polynomial or as a "right" polynomial.) By convention, 0 has degree $-\infty$. For every $n \in \mathbb{Z}_+$, we shall write $D_{\leq n}$ to denote the (A, A)-bimodule of Ore polynomials of degree $\leq n$. Define the isomorphism (of right A-modules) $\Phi_n : A^{\Delta(n)} \to D_{\leq n}$ by setting

$$\Phi_n(a) = \sum_{|i| \leq n} \partial^i a(i).$$

(If W is a set, we mean by $W^{\Delta(n)}$ the set of all W-valued functions defined on $\Delta(n)$.)

2. Tensor products over noncommutative rings

For convenience of the reader, we collect here the basic facts about tensor products over noncommutative rings (with 1).

If R is a (not necessarily commutative) ring, then given a right R-module M and left R-module N, one defines the tensor product $M \otimes_R N$ to be the abelian group generated by the symbols $m \otimes n$ ($m \in M$, $n \in N$) that satisfy the following relations only

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n,$$

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2,$$

$$mr \otimes n = m \otimes rn \;\; (\forall r \in R).$$

A bimodule is an abelian group that is both a left and a right module, such that the left and right multiplications are compatible. More precisely, if R and S are two rings, then an (R, S)-bimodule is an abelian group M such that M is a left R-module and a right S-module, and $r(ms) = (rm)s$ for all $r \in R$, $m \in M$, $s \in S$.

If $f : R \to S$ is a ring homomorphism, then S can be considered as a left R-module with the action $r \cdot s = f(r)s$, and with respect to this action S becomes an (R, S)-bimodule.

Notice that if R is a commutative ring, then a left (resp., right) R-module M can be given the structure of a right (resp., left) R-module by defining $mr = rm$ (resp., $rm = mr$). And this makes M into an (R, R)-bimodule.

The notion of bimodule allows us to view the tensor product as a module instead of merely as an abelian group. If M is a right R-module and N is an (R, S)-bimodule, then the tensor product $M \otimes_R N$ has the structure of a *right* S-module, where $(m \otimes n)s = m \otimes ns$. Likewise, if N is an (R, S)-bimodule and P is a left S-module, then the tensor product $N \otimes_S P$ has the structure of a *left* R-module, where $r(n \otimes p) = rn \otimes p$.

One can view R as an (R, R)-bimodule, and for any left R-module N,

$$R \otimes_R N \simeq N \tag{2}$$

as left R-modules.

The following lemma expresses the associativity property of the tensor product.

Lemma 1. *Suppose that M is a right R-module, N is an (R, S)-bimodule and P is a left S-module. Then there is a canonical isomorphism*

$$(M \otimes_R N) \otimes_S P \simeq M \otimes_R (N \otimes_S P)$$

of abelian groups.

The following lemma states that tensor product is right exact.

Lemma 2. *If*

$$M_1 \to M \to M_2 \to 0$$

is an exact sequence of right R-modules, then for any left R-module N

$$M_1 \otimes_R N \to M \otimes_R N \to M_2 \otimes_R N \to 0$$

is an exact sequence of abelian groups.

3. Jordan pencils

A pencil (over A) is a diagram

$$X \overset{K_1,\ldots,K_r,L}{\longrightarrow} Y,$$

where X, Y are finitely generated projective A-modules and $K_1, \ldots, K_r, L$ are linear maps from X to Y. It defines a homomorphism

$$X \otimes_A D \to Y \otimes_A D$$

given by

$$x \otimes f \mapsto \; K_1(x) \otimes \partial_1 f + \cdots + K_r(x) \otimes \partial_r f - L(x) \otimes f.$$

(This is a right module homomorphism over D.)

Jordan pencils are the simplest (non-trivial) examples of pencils, and we introduce them as follows.

Let n be a nonnegative integer. Set

$$X^{(n)} = \{ \begin{bmatrix} a_1 \\ \vdots \\ a_r \end{bmatrix} \in (A^{\Delta(n)})^r \mid \partial_1 \Phi_n(a_1) + \cdots + \partial_r \Phi_n(a_r) \in D_{\le n} \}.$$

For each $1 \le i \le r$, define a linear map $K_i^{(n)} : X^{(n)} \to A^{\Delta(n)}$ by

$$\begin{bmatrix} a_1 \\ \vdots \\ a_r \end{bmatrix} \mapsto a_i;$$

define $L^{(n)} : X^{(n)} \to A^{\Delta(n)}$ by

$$\begin{bmatrix} a_1 \\ \vdots \\ a_r \end{bmatrix} \mapsto \Phi_n^{-1}(\partial_1 \Phi_n(a_1) + \cdots + \partial_r \Phi_n(a_r)).$$

Lemma 3. *We have an exact sequence of A-modules*

$$0 \to X^{(n)} \to A^{\Delta(n)} \otimes_A D_{\le 1} \to D_{\le n+1} \to 0.$$

Proof. Left to the reader. □

It is immediate from this lemma that $X^{(n)}$ is a (finitely generated) projective A-module, and thus

$$X^{(n)} \overset{K_1^{(n)},\ldots,K_r^{(n)},L^{(n)}}{\longrightarrow} A^{\Delta(n)}$$

is a pencil. We call this the Jordan pencil of size n.

The following example explains why the Jordan pencils are named after Jordan.

Example. Assume that $r = 1$. Then $\Delta(n) = \{0, 1, \ldots, n\} = [0, n]$, and we have:

$$X^{(n)} = \{a \in A^{[0,n]} \mid a(n) = 0\} = A^{[0,n-1]}.$$

One can see that

$$K_1^{(n)} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad L^{(n)} = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}.$$

Remark that D can be considered as an (A, D)-bimodule by means of the canonical embedding $A \to D$; we also remark that $A^{\Delta(n)} \otimes_A D = D^{\Delta(n)}$. Define

$$A^{\Delta(n)} \otimes_A D \to D$$

to be the right module homomorphism given by

$$a \otimes f \mapsto \; \Phi_n(a) f.$$