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# Approximate and approximate null-controllability of a class of piecewise linear Markov switch systems

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ABSTRACT

We propose an explicit, easily-computable algebraic criterion for approximate null-controllability of a class of general piecewise linear switch systems with multiplicative noise. This gives an answer to the general problem left open in Goreac and Martinez (2015). The proof relies on recent results in Confortola et al. (2015) allowing to reduce the dual stochastic backward system to a family of ordinary differential equations. Second, we prove by examples that the notion of approximate controllability is strictly stronger than approximate null-controllability. A sufficient criterion for this stronger notion is also provided. The results are illustrated on a model derived from repressed bacterium operon (given in Krishna et al. (2005) and reduced in Crudu et al. (2009)).



## 1. Introduction

This short paper aims at giving an answer to an approximate (null-)controllability problem left open in [1]. We deal with Markovian systems of switch type consisting of a couple mode/trajectory denoted by $(\Gamma, X)$. The mode component $\Gamma$ evolves as a pure jump Markov process and cannot be controlled. It corresponds to spikes inducing regime switching. The second component $X$ obeys a controlled linear stochastic differential equation (SDE) with respect to the compensated random measure associated to $\Gamma$. The linear coefficients governing the dynamics depend on the current mode.

The controllability problem deals with criteria allowing one to drive the $X_T$ component arbitrarily close to acceptable targets. An extensive literature on controllability is available in different frameworks: finite-dimensional deterministic setting (Kalman's condition, Hautus test [2]), infinite dimensional settings (via invariance criteria in [3–7], etc.), Brownian-driven control systems (exact terminal-controllability in [8], approximate controllability in [9,10], mean-field Brownian-driven systems in [11], infinite-dimensional setting in [12–15], etc.), jump systems ([16,1], etc.). We refer to [1] for more details on the literature as well as applications one can address using switch models.

The paper [1] provides some necessary and some sufficient conditions under which approximate controllability towards null target can be achieved. In all generality, the conditions are either too strong (sufficient) or too weak (only necessary). Equivalence is obtained in [1] for particular cases: (i) Poisson-driven systems with mode-independent coefficients and (ii) continuous switching. In the present paper, we extend the work of [1] and give explicit equivalence criterion for the general switching case. The approach relies, in a first step, as it has already been the case in [1, Theorem 1], on duality techniques (briefly presented in Section 2.1). However, the intuition on this new criterion and its proof are extensively based on the recent ideas in [17]. The dual backward stochastic system associated to controllability is interpreted as a system of (backward) ordinary differential equations in Proposition 12. Reasoning on this new system provides the necessary and sufficient criterion for approximate null-controllability for general switching systems with mode-dependent multiplicative noise (Theorem 6 whose proof relies on Propositions 13 and 14). As a by-product, we considerably simplify the proofs of [1, Criteria 3 and 4] (in Section 2.3). Second, we give some elements on the stronger notion of (general) approximate controllability. While the notions of approximate and approximate null-controllability are known to coincide for Poisson-driven systems with mode-independent coefficients, we give an example (Example 9) showing that this is no longer the case for general switching systems. Furthermore, we show that the condition exhibited in [1, Proposition 3] in connection to approximate null-controllability is actually sufficient for general approximate controllability (see Condition 10). The proof follows, once again, from the deterministic reduction inspired by [17]. The theoretical results are illustrated on a model derived from repressed bacterium operon (given in [18] and reduced in [19]).

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We begin with presenting the problem, the standing assumptions and the main results: the duality abstract characterization in Theorem 2, the explicit criterion in Theorem 6. We give a considerably simplified proof of the results in [1] in Section 2.3. We discuss the difference between null and full approximate controllability in Section 2.4, Example 9 and give a sufficient criterion for the stronger notion of approximate controllability (Condition 10). Section 3 focuses on an example derived from [18] (see also [19]). The proofs of the results and the technical constructions allowing to prove Theorem 6 are gathered in Section 4.

## 2. The control system and main results

We briefly recall the construction of a particular class of pure jump, non explosive processes on a space $\Omega$ and taking their values in a metric space $(E, \mathcal{B}(E))$. Here, $\mathcal{B}(E)$ denotes the Borel $\sigma$-field of $E$. The elements of the space $E$ are referred to as modes. These elements can be found in [20] in the particular case of piecewise deterministic Markov processes (see also [21]). To simplify the arguments, we assume that $E$ is finite and we let $p \geq 1$ be its cardinal. The process is completely described by a couple $(\lambda, Q)$, where $\lambda : E \longrightarrow \mathbb{R}_+$ and the measure $Q : E \longrightarrow \mathcal{P}(E)$, where $\mathcal{P}(E)$ stands for the set of probability measures on $(E, \mathcal{B}(E))$ such that $Q(\gamma, \{\gamma\}) = 0$. Given an initial mode $\gamma_0 \in E$, the first jump time satisfies $\mathbb{P}^{0,\gamma_0}(T_1 \geq t) = \exp(-t\lambda(\gamma_0))$. The process $\Gamma_t := \gamma_0$, on $t < T_1$. The post-jump location $\gamma^1$ has $Q(\gamma_0, \cdot)$ as conditional distribution. Next, we select the inter-jump time $T_2 - T_1$ such that $\mathbb{P}^{0,\gamma_0}\left(T_2 - T_1 \geq t / T_1, \gamma^1\right) = \exp\left(-t\lambda\left(\gamma^1\right)\right)$ and set $\Gamma_t := \gamma^1$, if $t \in [T_1, T_2)$. The post-jump location $\gamma^2$ satisfies $\mathbb{P}^{0,\gamma_0}\left(\gamma^2 \in A / T_2, T_1, \gamma^1\right) = Q\left(\gamma^1, A\right)$, for all Borel set $A \subset E$. And so on. To simplify arguments on the equivalent ordinary differential system, following [17, Assumption (2.17)], we will assume that the system stops after a non-random, fixed number $M > 0$ of jumps i.e. $\mathbb{P}^{0,\gamma_0}(T_{M+1} = \infty) = 1$. The reader is invited to note (see Remark 5) that, for large $M$, the criteria given in the main result (Theorem 6) no longer depend on $M$ (due to the finite dimension of the mode and state spaces).

We look at the process $\Gamma$ under $\mathbb{P}^{0,\gamma_0}$ and denote by $\mathbb{F}^0$ the filtration $\left(\mathcal{F}_{[0,t]} := \sigma\{\Gamma_r : r \in [0,t]\}\right)_{t\geq 0}$. The predictable $\sigma$-algebra will be denoted by $\mathcal{P}^0$ and the progressive $\sigma$-algebra by $Prog^0$. As usual, we introduce the random measure $q$ on $\Omega \times (0,\infty) \times E$ by setting $q(\omega, A) = \sum_{k\geq 1} 1_{\left(T_k(\omega), \Gamma_{T_k(\omega)}(\omega)\right)\in A}$, for all $\omega \in \Omega$, $A \in \mathcal{B}(0,\infty) \times \mathcal{B}(E)$. The compensated martingale measure is denoted by $\widetilde{q}$. (For our readers familiar with [1], we emphasize that the notation is slightly different, the counting measure $q$ corresponds to $p$ in the cited paper and the martingale measure $\widetilde{q}$ replaces $q$ in the same reference. Further details on the compensator are given in Section 4.1.)

We consider a switch system given by a process $(X(t), \Gamma(t))$ on the state space $\mathbb{R}^N \times E$, for some $N \geq 1$ and the family of modes $E$. The control state space is assumed to be some Euclidian space $\mathbb{R}^d$, $d \geq 1$. The component $X(t)$ follows a controlled differential system depending on the hidden variable $\gamma$. We will deal with the following model ($A$ is implicitly assumed to be 0 after the last jump).

$$dX_s^{x,u} = \left[A(\Gamma_s) X_s^{x,u} + Bu_s\right] ds + \int_E C(\Gamma_{s-}, \theta) \times X_{s-}^{x,u} \widetilde{q}(ds, d\theta), \quad s \geq 0, \qquad X_0^{x,u} = x. \tag{1}$$

The operators $A(\gamma) \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times d}$ and $C(\gamma, \theta) \in \mathbb{R}^{N\times N}$, for all $\gamma, \theta \in E$. For linear operators, we denote by ker their kernel and by Im the image (or range) spaces. Moreover, the control process $u : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^d$ is an $\mathbb{R}^d$-valued, $\mathbb{F}^0$-progressively measurable, locally square integrable process. The space of all such processes will be denoted by $\mathcal{U}_{ad}$ and referred to as the family of admissible control processes. The explicit structure of such processes can be found in [22, Proposition 4.2.1], for instance. Since the control process does not (directly) intervene in the noise term, the solution of the above system can be explicitly computed with $\mathcal{U}_{ad}$ processes instead of the (more usual) predictable processes.

### 2.1. The duality abstract characterization of approximate null-controllability

We begin with recalling the following approximate controllability concepts.

**Definition 1.** The system (1) is said to be approximately controllable in time $T > 0$ starting from the initial mode $\gamma_0 \in E$, if, for every $\mathcal{F}_{[0,T]}$-measurable, square integrable $\xi \in \mathbb{L}^2\left(\Omega, \mathcal{F}_{[0,T]}, \mathbb{P}^{0,\gamma_0}; \mathbb{R}^N\right)$, every initial condition $x \in \mathbb{R}^N$ and every $\varepsilon > 0$, there exists some admissible control process $u \in \mathcal{U}_{ad}$ such that $\mathbb{E}^{0,\gamma_0}\left[\left|X_T^{x,u} - \xi\right|^2\right] \leq \varepsilon$. The system (1) is said to be approximately null-controllable in time $T > 0$ if the previous condition holds for $\xi = 0$ ($\mathbb{P}^{0,\gamma_0}$-a.s.).

At this point, let us consider the backward (linear) stochastic differential equation

$$\begin{cases} dY_t^{T,\xi} = \Big[-A^*(\Gamma_t) Y_t^{T,\xi} - \displaystyle\int_E \left(C^*(\Gamma_t, \theta) + I\right) Z_t^{T,\xi}(\theta) \\ \quad \lambda(\Gamma_t) Q(\Gamma_t, d\theta)\Big] dt + \displaystyle\int_E Z_t^{T,\xi}(\theta)\, q(dt, d\theta), \\ Y_T^{T,\xi} = \xi \in \mathbb{L}^2\left(\Omega, \mathcal{F}_{[0,T]}, \mathbb{P}^{0,\gamma_0}; \mathbb{R}^N\right). \end{cases} \tag{2}$$

Classical arguments on the controllability operators and the duality between the concepts of controllability and observability lead to the following characterization (cf. [1, Theorem 1]).

**Theorem 2** (*[1, Theorem 1]*). *The necessary and sufficient condition for approximate null-controllability (resp. approximate controllability) of (1) with initial mode $\gamma_0 \in E$ is that any solution $\left(Y_t^{T,\xi}, Z_t^{T,\xi}(\cdot)\right)$ of the dual system (2) for which $Y_t^{T,\xi} \in \ker B^*$, $\mathbb{P}^{0,\gamma_0} \otimes Leb$ almost everywhere on $\Omega \times [0,T]$ should equally satisfy $Y_0^{T,\xi} = 0$, $\mathbb{P}^{0,\gamma_0}$-almost surely (resp. $Y_t^{T,\xi} = 0$, $\mathbb{P}^{0,\gamma_0} \otimes Leb - a.s.$).*

**Remark 3.** Concerning the operator $A$, it is assumed to be a switched matrix but it could also depend on $(t, \Gamma_t)$ or on all the times and marks prior to $t$. This is why, we implicitly assumed that $A = 0$ after the last jump ($M$th) occurs. Similar assertions are true for $C$ (otherwise, the backward equation (2) should be written with the compensator $\widehat{q}$ replacing $\lambda(\Gamma_t) Q(\Gamma_t, d\theta)$.) The reader may also look at the end of Section 4.1.

### 2.2. Main result: an iterative invariance criterion

Before stating the main result of our paper, we need the following invariance concepts (cf. [4,3]).

**Definition 4.** We consider a linear operator $\mathcal{A} \in \mathbb{R}^{N\times N}$ and a family $\mathcal{C} = (\mathcal{C}_i)_{1\leq i\leq k} \subset \mathbb{R}^{N\times N}$.

(i) A set $V \subset \mathbb{R}^N$ is said to be $\mathcal{A}$-invariant if $\mathcal{A}V \subset V$.
(ii) A set $V \subset \mathbb{R}^N$ is said to be $(\mathcal{A}; \mathcal{C})$-invariant if $\mathcal{A}V \subset V + \sum_{i=1}^k \operatorname{Im} \mathcal{C}_i$.

We construct a mode-indexed family of linear subspaces of $\mathbb{R}^N$ denoted by $\left(V_\gamma^{M,n}\right)_{0\leq n\leq M, \gamma\in E}$ by setting

$$\mathcal{A}^*(\gamma) := A^*(\gamma) - \int_E \left(C^*(\gamma, \theta) + I\right) \lambda(\gamma) Q(\gamma, d\theta) \quad \text{and}$$

$$V_\gamma^{M,M} = \ker B^*, \tag{3}$$