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Proper representations of (multivariate) linear differential systems

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ABSTRACT

A proper representation of a linear differential system is a representation with no singularity at infinity. It is shown that such a representation always exists. It turns out that for proper representations having minimal number of rows is equivalent to having minimal total row degree. One is led therefore to a natural definition of the notion of minimality. What is remarkable is that a minimal proper representation is uniquely determined up to premultiplication by a unimodular polynomial matrix of special form. This uniqueness result allows, in particular, to introduce important integer invariants.

1. Introduction

Throughout, $\mathbb{F}$ is the field of real or complex numbers, n and q are fixed positive integers, $s = (s_1, \ldots, s_n)$ is a sequence of indeterminates, and s_0 is an extra ("homogenizing") indeterminate. We let $S = \mathbb{F}[s]$ and $T = \mathbb{F}[s_0, s]$, and denote by $\mathcal{U}$ the space of C^∞-functions (or distributions) defined on some domain of $\mathbb{R}^n$.

Proper polynomial matrices are polynomial matrices (with entries in S) that behave well at infinity. They play a significant role in the classical one-dimensional linear systems theory, and we claim that their role in higher dimensions must be analogous. (The infinity is the complement of the affine space $\mathbb{A}^n$ to the projective space $\mathbb{P}^n$, that is, the hyperplane in $\mathbb{P}^n$ defined by the equation $s_0 = 0$.)

Assume that we have a linear time-invariant (LTI) differential system $\mathcal{B} \subseteq \mathcal{U}^q$, and assume that it is represented by a polynomial matrix $R \in S^{p\times q}$, so that

$$\mathcal{B} = KerR(\partial).$$

As is well-known, the submodule $R^{tr}S^p \subseteq S^q$ is independent of the choice of R and is an intrinsic invariant of $\mathcal{B}$; moreover, by Oberst's duality, this is a full invariant. There is a procedure, called *homogenization* (and denoted here by the superscript "h"), that produces homogeneous things from non-homogeneous ones. Homogenizing the submodule $R^{tr}S^p \subseteq S^q$, we get a homogeneous submodule $(R^{tr}S^p)^h \subseteq T^q$. Like $R^{tr}S^p$, this module also is an intrinsic full invariant. Alternatively, one can homogenize first R and then take the homogeneous submodule $(R^h)^{tr}T^p \subseteq T^q$. The latter, however, depends on R and is *not* an invariant of $\mathcal{B}$. One has

$$(R^h)^{tr}T^p \subseteq (R^{tr}S^p)^h.$$

The equality holds if and only if s_0 is not a zero divisor on the quotient module $T^q/(R^h)^{tr}T^p$. In our opinion, representations having this property are of primary importance, and we call them proper.

We think that it is not proper to represent an LTI differential system via an improper polynomial matrix since it does not provide an adequate description at infinity.

Remark. As explained in the concluding section, properness should be interpreted as the property of "controllability at infinity".

In this paper, we prove that proper representations always exist. Next, we show that for proper representations there is a good notion of minimality. Namely, we show that if R is a proper representation of an LTI differential system $\mathcal{B}$, then the following two conditions are equivalent:

(a) R has the minimum possible number of rows (among all proper representations of $\mathcal{B}$);
(b) R has the minimum possible total row degree (among all proper representations of $\mathcal{B}$).

Proper representations satisfying these conditions are called minimal. The uniqueness result that we prove states that minimal proper representations are uniquely determined up to, the so-called, Brunovsky equivalence.

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Remark. As is well-known, in dimension 1, an LTI differential system has a full row rank proper representation, which certainly is a minimal proper representation. This is unfortunately not the case in higher dimensions; even more, an LTI differential system may have a full row rank representation, but not a full row rank proper representation (see Example 10 in Section 6).

For every $d \in \mathbb{Z}$, we shall write $S_{\leq d}$ to denote the space of polynomials (in S) of degree $\leq d$ and T_d for the space of homogeneous polynomials (in T) of degree d. It is worth noting that $S_{\leq d} = \{0\}$ and $T_d = \{0\}$ for negative d. (In Section 6, we shall need homogeneous polynomials in S as well, and S_d will stand for the space of all homogeneous polynomials that have degree d.) For a positive integer p, we write $[1, p]$ for the set $\{1, \dots, p\}$.

This article can be viewed as an attempt to generalize Section X in Willems [1] to higher dimensions. We remark also that much of material presented here is adapted from [2] (which, in turn, is based on [3]).

2. Preliminaries on graded and filtered modules

Powerful tools for the study of LTI differential systems are S-modules. But S-modules disregard the infinity, and therefore are useless when one wants to carry out the study at infinity. Graded T-modules have the advantage that they allow to study LTI differential systems (simultaneously) both on the finite domain and at the infinity.

A graded module over T is a module M together with a gradation, i.e., a sequence $M_0, M_1, M_2, \dots$ of $\mathbb{F}$-linear subspaces of M such that

$$M = \bigoplus_{d\geq 0} M_d \quad \text{and} \quad s_k M_d \subseteq M_{d+1} \quad \forall k, d.$$

(For $d < 0$, one puts $M_d = \{0\}$.) The elements of M_d are called the homogeneous elements of M of degree d. A submodule $N \subseteq M$ is called a graded submodule of M if $N = \bigoplus (N \cap M_d)$.

For a graded T-module M and a nonnegative integer k, one denotes by $M(-k)$ the graded T-module whose homogeneous components are defined by

$$M(-k)_d = M_{d-k}.$$

Example 1. Let p be a positive integer. Then, a function $a : [1, p] \to \mathbb{Z}_+$ determines on T^p a gradation consisting of the spaces

$$T^p(-a)_d = \{f \in T^p | \deg(f_i) = d - a(i)\} \quad (d \geq 0).$$

The module T^p equipped with this gradation is denoted by $T^p(-a)$. Notice that

$$T^p(-a) = T(-a(1)) \oplus \cdots \oplus T(-a(p)).$$

A homomorphism of graded modules $M \to N$ is a module homomorphism $u : M \to N$ such that $u(M_d) \subseteq N_d$ for all $d \geq 0$.

Example 2. Let a and b be nonnegative integers. Homomorphisms from $T(-a)$ to $T(-b)$ are exactly multiplications by homogeneous polynomials of degree $a - b$. That is,

$$Hom(T(-a), T(-b)) = T_{a-b}.$$

A polynomial matrix with entries in T is called column-homogeneous if all the entries in each column are homogeneous and have the same degree.

Example 3. A column-homogeneous polynomial matrix H of size $q \times p$ and with column degree function a determines a homomorphism of graded modules

$$H : T^p(-a) \to T^q.$$

The homogenization in degree d is the bijective linear map $\theta_d : S_{\leq d} \to T_d$ defined by the formula

$$\theta_d(f) = s_0^d f(s/s_0).$$

(Here and below s/s_0 stands for $(s_1/s_0, \dots, s_n/s_0)$.)

Example 4. Let $n = 2$ and $f = 2s_1^3 s_2 + 1$. Then

$$\theta_4(f) = 2s_1^3 s_2 + s_0^4 \quad \text{and} \quad \theta_5(f) = 2s_0 s_1^3 s_2 + s_0^5.$$

If $A \subseteq S^q$ is a submodule, the homogenization A^h of A is defined to be

$$A^h = \bigoplus_{d\geq 0} A_d^h,$$

where $A_d^h = \theta_d(A_{\leq d})$. This is the smallest graded submodule of T^q that contains A.

The dehomogenization is the operator $T \to S$ defined by

$$u(s_0, s) \mapsto u(1, s).$$

It is worth noting that if $d \geq 0$, then

$$\begin{aligned} &\forall u \in T_d, \quad \theta_d(u(1, s)) = u \quad \text{and} \\ &\forall f \in S_{\leq d}, \quad (\theta_d f)(1, s) = f. \end{aligned} \tag{1}$$

If $B \subseteq T^q$ is a graded submodule, the dehomogenization B^{dh} of B is its image under the dehomogenization operator, i.e.,

$$B^{dh} = \{u(1, s) | \ u \in B\}.$$

This is a submodule of S^q.

We pass now to filtered S-modules, which are more natural tools than graded T-modules. (However, graded modules are superior from the purely technical point of view.) The point is that the modules associated with LTI differential systems have the structure of a filtered S-module.

Let M be a module over S. A filtration on M is an ascending chain

$$M_{\leq 0} \subseteq M_{\leq 1} \subseteq M_{\leq 2} \subseteq \cdots$$

of linear subspaces of M such that

$$M = \bigcup_{d\geq 0} M_{\leq d} \quad \text{and} \quad s_k M_{\leq d} \subseteq M_{\leq d+1} \quad \forall k, d.$$

A module with a filtration is called a filtered module. A submodule N of a filtered module M is a filtered module with the filtration

$$N_{\leq d} = N \cap M_{\leq d}, \quad d \geq 0.$$

(For $d < 0$, put $M_{\leq d} = \{0\}$.) If M is a filtered module and k a nonnegative integer, we denote by $M[-k]$ the filtered module with the filtration defined by

$$M[-k]_{\leq d} = M_{\leq d-k}.$$

Example 5. Let p be a positive integer. Then, a function $a : [1, p] \to \mathbb{Z}_+$ determines on S^p a filtration consisting of the spaces

$$S^p[-a]_{\leq d} = \{f \in S^p | \deg(f_i) \leq d - a(i)\} \quad (d \geq 0).$$

The module S^p equipped with this filtration is denoted by $S^p[-a]$. Notice that

$$S^p[-a] = S[-a(1)] \oplus \cdots \oplus S[-a(p)].$$

A homomorphism of filtered modules $M \to N$ is a module homomorphism $u : M \to N$ such that

$$\forall d \geq 0, \quad u(M_{\leq d}) \subseteq N_{\leq d}.$$

Notice that $Ker(\varphi)$ is a graded submodule of M and $Im(\varphi)$ is a graded submodule of N.