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Characterization of maximum hands-off control

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ABSTRACT

Maximum hands-off control aims to maximize the length of time over which zero actuator values are applied to a system when executing specified control tasks. To tackle such problems, recent literature has investigated optimal control problems which penalize the size of the support of the control function and thereby lead to desired sparsity properties. This article gives the exact set of necessary conditions for a maximum hands-off optimal control problem using an $\mathcal{L}_0$-norm, and also provides sufficient conditions for the optimality of such controls. Numerical example illustrates that adopting an $\mathcal{L}_0$ cost leads to a sparse control, whereas an $\mathcal{L}_1$-relaxation in singular problems leads to a non-sparse solution.

1. Introduction

Motivated by a diverse array of applications in automotive industry, railway vehicles, and networked control, the recent works [1,2] dealt in detail with the concept of *maximum hands-off control*. The purpose of maximum hands-off control is to design actuator signals which are most often zero, but nonetheless achieve given control objectives. This motivates the use of instantaneous cost functions where the control effort is penalized via the $\mathcal{L}_0$-"norm", thereby leading to a *sparse* control function, cf. [3–9]. Sparse controls are of great importance in situations where a central processor must be shared by different controllers, and sparse control is a new and emerging area of research, including applications in the theory of control of partial differential equations [10–14].

Due to the discontinuous and non-convex nature of the instantaneous cost function in $\mathcal{L}_0$-optimal control problems, solving such problems is in general difficult. Hence, the precursor article [2] focused on relaxations to the problem, akin to methods used in compressed sensing applications [15]. To be more precise, [2] examined smooth and convex relaxations of the maximum hands-off control problem, including considering an $\mathcal{L}_1$-cost and regularizations with an $\mathcal{L}_2$-cost to obtain smooth hands-off control. (It is a well-known and classical result that under "nonsingularity" assumptions on the control system [16, Chapter 8], $\mathcal{L}_1$-costs lead to sparse solutions in the control. However, in singular problem instances, it is unclear whether $\mathcal{L}_1$-regularizations lead to sparse solutions.) The exact $\mathcal{L}_0$-optimal control problem was not investigated in [2].

The purpose of the present article is to complement [2] by directly dealing with the underlying non-smooth and non-convex $\mathcal{L}_0$-optimal control problem without the aid of smooth or convex relaxations. We will focus on nonlinear controlled dynamical systems of the form

$$\dot{z}(t) = \phi\big(z(t), u(t)\big) \tag{1}$$

with state z, input u and where $\phi : \mathbb{R}^d \times \mathbb{R}^m \longrightarrow \mathbb{R}^d$ is a continuously differentiable map describing the open-loop system dynamics. The maximum hands-off control problem aims to minimize the support of the control map, or in other words, maximize the time duration over which the control map is exactly zero.

In other words, given real numbers $a, b \in \mathbb{R}$ with $a < b$, vectors $A, B \in \mathbb{R}^d$, a compact set $\mathbb{U} \subset \mathbb{R}^m$ containing $0 \in \mathbb{R}^m$ in its interior, we consider the optimal control problem

$$\begin{aligned} &\underset{u}{\text{minimize}} && \|u\|_{\mathcal{L}_0([a,b])} \\ &\text{subject to} && \begin{cases} \dot{z}(t) = \phi\big(z(t), u(t)\big) & \text{for a.e. } t \in [a,b], \\ z(a) = A, \quad z(b) = B, \\ u : [a,b] \longrightarrow \mathbb{U} & \text{Lebesgue measurable.} \end{cases} \end{aligned} \tag{2}$$

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Here the $\mathcal{L}_0$-"norm"[1] of a map $u : [a, b] \longrightarrow \mathbb{U}$ is defined by the Lebesgue measure of the support of u, i.e.,

$$\|u\|_{\mathcal{L}_0([a,b])} := \text{Leb}\Big(\big\{s \in [a, b] \,\big|\, u(s) \neq 0\big\}\Big).$$

Observe that if the minimum time to transfer the system states from A to B is larger than the given duration $b-a$, then the optimal control problem (2) has no solution. Thus, a standing assumption used throughout this work is that there is a feasible solution to (2). In other words, despite the limited control authority described by the compact set $\mathbb{U}$, we shall assume that it is possible to steer the system states from A to B in finite time $b - a$. Observe also that, unlike minimum attention control à la [17], the optimal control problem (2) does not penalize the rate of change of the control. Nonetheless, (2) can be viewed through the looking glass of least attention in the sense that the control is 'active' for the least duration of time. The current work investigates optimality in (2) using a nonsmooth maximum principle as summarized in [18, Chapter 22].

The main contributions and outline of this article are given below:

(i) We show that (2) can be recast in the form of an optimal control problem involving an integral cost with a discontinuous cost function. We apply a non-smooth Pontryagin maximum principle directly to problem (2) and obtain an exact set of necessary conditions for optimality. This result is presented in Section 2. It characterizes solutions to (2) provided that they exist.

(ii) Section 3 sheds further insight into the case where the system dynamics in (1) are linear. Section 4 illustrates that, perhaps contrary to intuition, in singular problem instances, $\mathcal{L}_1$-relaxations may fail to give sparse controls; cf. [16, Chapter 8].

(iii) For plant models that are linear in the states, we show in Section 5 that under normality of optimal state-action trajectories, the necessary conditions for optimality are also sufficient.

Notation. The notations employed in this article are standard. The Euclidean norm of a vector z, belonging to the d-dimensional Euclidean space $\mathbb{R}^d$, is denoted by $\|z\|$; vectors are treated as column vectors. For a set S we let $z \longmapsto \mathbb{1}_S(z)$ denote the indicator (characteristic) function of the set S defined to be 1 if $z \in S$ and 0 otherwise.

Remark 1. The version of the maximum hands-off control problem posed in [1,2] is slightly different from the one we examine in (2). Indeed, [2] studies the following problem:

$$\begin{aligned} &\underset{u}{\text{minimize}} \quad \frac{1}{b-a}\sum_{i=1}^{m} \lambda_i \|u_i\|_{\mathcal{L}_0([a,b])} \\ &\text{subject to} \quad \begin{cases} \dot z(t) = \phi\big(z(t), u(t)\big) & \text{for a.e. } t \in [a, b], \\ z(a) = A, \quad z(b) = B, \\ u : [a, b] \longrightarrow \mathbb{U} \quad \text{Lebesgue measurable}, \end{cases} \end{aligned} \tag{3}$$

where $\{\lambda_i\}_{i=1}^m$ are given positive weights. This cost function features the controls of a multivariable plant as additive terms. In contrast, and by noting that

$$\int_a^b \mathbb{1}_{\{0\}}(u(s))\,ds = \int_a^b \prod_{i=1}^{m} \mathbb{1}_{\{0\}}(u_i(s))\,ds,$$

(where the 0 on the left-hand side belongs to $\mathbb{R}^m$ and the one on the right-hand side belongs to $\mathbb{R}$,) the cost function (2) features a multiplicative form in the controls. The techniques exposed for (2) in the sequel carry over in a straightforward fashion to (3). In order not to blur the message of this article, we stick to the simpler case of (2). □

2. Necessary conditions for optimality

By definition, we have

$$\|u\|_{\mathcal{L}_0([a,b])} = b - a - \int_a^b \mathbb{1}_{\{0\}}(u(s))\,ds. \tag{4}$$

Since a and b are fixed, the minimization of $\|u\|_{\mathcal{L}_0([a,b])}$ in (2) is equivalent to the minimization of $-\int_a^b \mathbb{1}_{\{0\}}(u(s))\,ds$. In view of this, we rewrite the optimal control problem (2) as

$$\begin{aligned} &\underset{u}{\text{minimize}} \quad -\int_a^b \mathbb{1}_{\{0\}}(u(s))\,ds \\ &\text{subject to} \quad \begin{cases} \dot z(t) = \phi\big(z(t), u(t)\big) & \text{for a.e. } t \in [a, b], \\ z(a) = A, \quad z(b) = B, \\ u : [a, b] \longrightarrow \mathbb{U} \quad \text{Lebesgue measurable}. \end{cases} \end{aligned} \tag{5}$$

We have the following proposition:

Proposition 1. *Associated to every solution $[a, b] \ni t \longmapsto \big(z_\star(t), u_\star(t)\big)$ to (2) there exist an absolutely continuous curve $[a, b] \ni t \longmapsto p(t) \in \mathbb{R}^d$ and a number $\eta = 0$ or 1 such that for a.e. $t \in [a, b]$:*

$$\begin{cases} \dot z_\star(t) = \phi\big(z_\star(t), u_\star(t)\big), \quad z_\star(a) = A, \quad z_\star(b) = B, \\ \dot p(t) = -\Big(\partial_z \phi\big(z_\star(t), u_\star(t)\big)\Big)^\top p(t), \\ u_\star(t) \in \underset{v \in \mathbb{U}}{\arg\max}\Big\{\big\langle p(t), \phi\big(z_\star(t), v\big)\big\rangle + \eta \mathbb{1}_{\{0\}}(v)\Big\}, \end{cases} \tag{6}$$

and

$$\big(\eta, p(t)\big) \neq (0, 0) \in \mathbb{R} \times \mathbb{R}^d \quad \text{for all } t \in [a, b]. \tag{7}$$

A proof of Proposition 1 is provided in the Appendix.

Remark 2. Proposition 1 gives a set of necessary conditions for optimality of state-action trajectories $t \longmapsto \big(z_\star(t), u_\star(t)\big)$ in the same spirit as the standard first order necessary conditions for an optimum in a finite-dimensional optimization problem. We see that the ordinary differential equations (o.d.e.'s) describing the system state $z_\star$ and its adjoint p constitute a set of $2d$-dimensional o.d.e.'s with $2d$ constraints. This amounts to a well-defined boundary value problem in the sense of Carathéodory [19, Chapter 1]. Indeed, the control map $u_\star$ is Lebesgue measurable, and depends parametrically on p; therefore, the right-hand side of (1) under $u_\star$ satisfies the Carathéodory conditions [19, Chapter 1] that guarantee existence of a Carathéodory solution.

Remark 3. Numerical solutions to differential equations such as the ones in (6) are typically carried out by what are known as the shooting and multiple shooting methods. This is an active area of research; see [20, Chapter 3] for a detailed discussion.

Remark 4. The quadruple $\big(\eta, p(\cdot), z_\star(\cdot), u_\star(\cdot)\big)$ is known as the *extremal lift* of the optimal state-action trajectory $\big(z_\star(\cdot), u_\star(\cdot)\big)$. The scalar η is known as the *abnormal multiplier*. If $\eta = 1$, then the extremal $t \longmapsto \big(\eta, p(t), z_\star(t), u_\star(t)\big)$ is said to be normal; if $\eta = 0$, then the extremal is said to be abnormal. The scalar η is a Lagrange multiplier associated to the instantaneous cost. Interestingly, the curves for which $\eta = 0$ are not detected by the

[1] Note that our convention of calling the map $u \longmapsto \|u\|_{\mathcal{L}_0([a,b])}$ a "norm" is technically not precise because this map does not satisfy the positive homogeneity property despite being positive definite and satisfying the triangle inequality. However, here we prefer to overload the word 'norm' for brevity.