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# Reduction model approach for linear time-varying systems with input delays based on extensions of Floquet theory

Frédéric Mazenc[a,*], Michael Malisoff[b]

[a] EPI DISCO INRIA-Saclay, L2S, CNRS CentraleSupelec, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France

[b] Department of Mathematics, 303 Lockett Hall, Louisiana State University, Baton Rouge, LA 70803-4918, USA

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ABSTRACT

We solve stabilization problems for linear time-varying systems under input delays. We show how changes of coordinates lead to systems with time invariant drifts, which are covered by the reduction model method and which lead to the problem of stabilizing a time-varying system without delay. For continuous time periodic systems, we can use Floquet theory to find the changes of coordinates. We also prove an analogue for discrete time systems, through a discrete time extension of Floquet theory.



## 1. Introduction

This note continues our search (begun in [1,2]) for extensions of the classical reduction model method that cover time-varying systems with input delays. Input delays are common when controllers are remotely implemented; see [3–8] for more motivation. The reduction method has its origins in the works [9–11] by Artstein and others, who focused on continuous time invariant linear systems.

Stabilization problems for linear time-varying systems with delays have been studied in fewer works. In most of them, time-varying Lyapunov functions are needed; see for instance [12,13] for the use of strict Lyapunov functions, and [14] for Razumikhin–Lyapunov functions. One useful Lyapunov-based approach to delay systems entails solving the stabilization problem with the input delay set equal to zero, and then using Lyapunov–Krasovskii functionals to look for upper bounds on the input delays that the closed loop system can tolerate without sacrificing the stability performance; see [15,16]. Linear time-varying systems arise in the context of the local stabilization of a trajectory of a nonlinear system, but are beyond the scope of the classical reduction model method. The main differences between reduction approaches and the Lyapunov–Krasovskii functional approaches such as those in [16] are that (a) under certain delay bounds, methods such as [16] lead to relatively simple controllers that do not require the distributed terms that are used in reduction model methods and (b) reduction model methods usually make it possible to compensate for arbitrarily long input delays, by using the delay value in the dynamic feedback control design.

Our work [2] extends the reduction model method to linear time-varying systems, using two approaches. One approach in [2] leads to a control formula that involves the fundamental matrix for the corresponding uncontrolled system (i.e., the time-varying system obtained from the original system by setting the input equal to zero in the original system), and so may be difficult to apply in practice. The other control design in [2] does not require a formula for the fundamental matrix, but requires that the input delay stay below a suitable constant bound. By contrast, [1] covers time-varying nonlinear systems whose nonlinear parts satisfy certain conditions, and then builds a reduction model control for the linearization of the system.

One natural research direction for addressing the challenges of extending the reduction model method to time-varying linear systems, and for analogous problems for discrete time systems, is to seek analogues of Floquet's theory; see [17, Section 3.5]. Floquet's theory covers systems without controls. One of its basic results is that if a time-varying linear system is periodic, then it can be transformed into a time invariant system through a periodic change of coordinates. This suggests the possibility of using Floquet theory to transform a time-varying linear control system into a new time-varying linear control system with a time invariant draft, and stabilizing the new system by the reduction approach.

* Corresponding author. Fax: +33 0169851765.
*E-mail addresses:* frederic.mazenc@l2s.centralesupelec.fr (F. Mazenc), malisoff@lsu.edu (M. Malisoff).

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One key observation in this work is that such a transformation can be done under periodicity of the coefficient matrices in the system, and that this simplifies the stabilization problem to one that involves globally asymptotically stabilizing systems with no delay. Our assumptions are novel. We also provide analogues for nonperiodic or discrete time systems.

Discrete time systems with delay are important because they can be used to model some engineering devices; see [18–21]. However, not many contributions are concerned with time-varying discrete time systems with delay. Our discrete time delayed systems in this work have the form

$$x_{k+1} = A_k x_k + B_k u_{k-r} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^p$ is the control, and $r \in \mathbb{N}$ is the delay. Here and in the sequel, the dimensions and delays are arbitrary. For the case of time invariant coefficients, the work [19] uses dynamic extensions to transform (1) into systems with no delay, in the special case of networked control systems. There are other stabilization results for communication systems that are based on state augmentation; see, e.g., [22,23] for results for time invariant systems based on linear matrix inequalities. See also [24] for a prediction based approach for (1) in the time invariant case. For time-varying continuous time systems with delay in the input, the reduction model approach can be applied under conditions pertaining to the speed of variation of the time-varying matrices; see [2]. However, to the best of our knowledge, no discrete time version of [2] exists. Also, [25] is concerned with time invariant systems.

We propose a rather general solution to the problem of exponential stabilization of (1) through the reduction model approach, including cases where the coefficient matrices are not necessarily periodic, with an arbitrarily large delay $r$. It decomposes into two steps. First, under reasonable assumptions, we transform (1) into a system that is autonomous when the control is set equal to zero. Then, we adapt the reduction method to the resulting dynamics, using a novel discrete time analogue of an operator that is used in the predictor based analysis in [25]. Our treatment of (1) also has implications for using reduction model controllers in continuous time systems, because in practice, implementing controllers in continuous time systems uses discretizations, leading to discrete time delay systems of the form (1). We illustrate our theory in two examples, including a discrete time linear system in which the coefficient matrices are not periodic.

## 2. Preliminary results in continuous time

We use the following notation and definitions. Let $|\cdot|$ be the usual Euclidean norm of matrices and vectors, $I_n$ be the $n \times n$ identity matrix, and $\mathbb{N} = \{1, 2, \ldots\}$. For any function $\phi : S \to \mathbb{R}^p$ that is defined on any subset $S$ of a Euclidean space, we use $|\phi|_J$ to denote its supremum over any set $J \subseteq S$. We often leave out the arguments of functions, when they are clear, and for matrix valued functions $E$ such that $E(t)$ is invertible for all $t$ in the domain of $E$, we use $E^{-1}(t)$ to mean the matrix inverse of the matrix $E(t)$ for all $t$.

### 2.1. Fundamental general result

Consider the system

$$\dot{x} = A(t)x + F(t)u(t-\tau) \tag{2}$$

where the state $x$ and the input $u$ are valued in $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively, the functions $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $F : \mathbb{R} \to \mathbb{R}^{n\times p}$ are continuous, and $\tau > 0$ is any positive constant delay. We introduce the following assumption. See below for ways to build the required function $P$.

**Assumption 1.** There exist a constant matrix $A_c \in \mathbb{R}^{n\times n}$, a $C^1$ function $P : \mathbb{R} \to \mathbb{R}^{n\times n}$ such that $P^{-1}(t)$ is defined for all $t$, and a constant $p_M > 0$ such that

$$|P(t)| + |P^{-1}(t)| \le p_M \tag{3}$$

and

$$\dot{P}(t) = A_c P(t) - P(t)A(t) \tag{4}$$

hold for all $t \ge 0$.

The intuition between Assumption 1 is that it encompasses key properties that hold for the special case of time invariant systems $\dot{x} = Mx + Fu(t-\tau)$ and that we will use to transform (2) into a new system that is time invariant when the input is 0; see Remark 1. For time invariant cases, we can satisfy Assumption 1 using the identity matrix $P(t) = I_n$ and $A_c = M$. In Section 2.2, we give general ways to satisfy Assumption 1 for time varying systems. We use the following key observations:

**Lemma 1.** *Assume that the system (2) satisfies Assumption 1. Then the time-varying change of coordinates*

$$z = P(t)x \tag{5}$$

*transforms (2) into*

$$\dot{z}(t) = A_c z(t) + P(t)F(t)u(t-\tau). \tag{6}$$

*Also, the operator*

$$Z(t) = z(t) + \int_{t-\tau}^{t} e^{A_c(t-m-\tau)} P(m+\tau)F(m+\tau)u(m)\mathrm{d}m \tag{7}$$

*transforms (6) into the system*

$$\dot{Z}(t) = A_c Z(t) + e^{-A_c\tau} P(t+\tau)F(t+\tau)u(t) \tag{8}$$

*with the state variable $Z$.*

**Proof.** Our choice (5) of $z$ gives $\dot{z}(t) = \dot{P}(t)x(t) + P(t)\dot{x}(t) = \dot{P}(t)x(t) + P(t)[A(t)x(t) + F(t)u(t-\tau)]$. Then (6) follows from (4) and our choice of $z$. Also, the time derivative of (7) along all solutions of (6) is

$$\begin{aligned}\dot{Z}(t) &= A_c z(t) + P(t)F(t)u(t-\tau) \\ &\quad + A_c \int_{t-\tau}^{t} e^{A_c(t-m-\tau)} P(m+\tau)F(m+\tau)u(m)\mathrm{d}m \\ &\quad + e^{-A_c\tau}P(t+\tau)F(t+\tau)u(t) - P(t)F(t)u(t-\tau) \\ &= A_c Z(t) + e^{-A_c\tau}P(t+\tau)F(t+\tau)u(t)\end{aligned} \tag{9}$$

which gives the second conclusion. □

Lemma 1 implies that if one knows a function $P$ that leads to a time invariant system when the input is zero, then in practice, we can use the reduction model approach to obtain a new system without delays.

**Remark 1.** Before discussing ways to find $P$, we remark that similar reasoning applies to systems

$$\dot{x} = A(t)x + \int_{t-\tau}^{t} F(\ell)u(\ell)\mathrm{d}\ell \tag{10}$$

with distributed delay in the input and continuous matrix valued functions $A$ and $F$. To see how, notice that if Assumption 1 is satisfied, and if we define $z$ by (5) as before and redefine $Z$ to be

$$Z(t) = z(t) + \int_{t-\tau}^{t} e^{A_c(t-m-\tau)}P(m+\tau) \times \left(\int_{m}^{t} F(\ell)u(\ell)\mathrm{d}\ell\right)\mathrm{d}m, \tag{11}$$