Systems & Control Letters 94 (2016) 111–117

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Output error minimizing back and forth nudging method for initial state recovery

Atte Aalto

Inria, Université Paris–Saclay, MΞDISIM team, 1 rue Honoré d'Estienne d'Orves, 91120 Palaiseau, France

ARTICLE INFO

Article history:
Received 15 October 2015
Received in revised form
20 April 2016
Accepted 3 June 2016
Available online 27 June 2016

Keywords:
Back and forth nudging
State estimation
Output error minimization
Observers
Variational data assimilation

ABSTRACT

We show that for linear dynamical systems with skew-adjoint generators, the initial state estimate given by the back and forth nudging method with colocated feedback, converges to the minimizer of the discrepancy between the measured and simulated outputs — given that the observer gains are chosen suitably and the system is exactly observable. If the system's generator A is essentially skew-adjoint and dissipative (with not too much dissipation), the colocated feedback has to be corrected by the operator $e^{At}e^{A^*t}$ in order to obtain such convergence. In some special cases, a feasible approximation for this operator can be found analytically. The case with wave equation with constant dissipation will be presented.

1. Introduction

This paper deals with the problem of retrieving the initial state of a – possibly infinite-dimensional – linear dynamical system from the noisy output measurements of the system over a given, finite time interval $[0, T]$. A more or less classical approach is to minimize the quadratic discrepancy between the measured and modeled outputs over all possible initial states. This approach is often called *variational data assimilation*—for details and references, see [1] by Le Dimet et al. and [2] by Teng et al. In the case of a linear system, this approach leads to a linear-quadratic optimization problem, whose solution amounts to computing and inverting the observability Gramian. This approach is seemingly simple, but when the system's dimension is high, the optimization task may be numerically challenging, so alternative methods are called for.

One alternative is the *back and forth nudging* (BFN) method, introduced by Auroux and Blum in [3,4]. The method is based on using a Luenberger observer alternately forward and backward in time over and over again. In these papers the theory is developed for finite-dimensional systems and it is assumed that the full state is observed. The generalization to infinite-dimensional systems and more general observation operators is presented by Ramdani et al. in [5]. There it is shown that in the absence of any noise terms, the BFN method converges exponentially to the true initial state. They assume that the system is exponentially stabilizable both to forward and backward directions. The BFN method is presented and reviewed in Section 2.

Whereas the variational method gives equal weight to all measurements on the time interval $[0, T]$, the BFN method emphasizes the measurements, and hence also the measurement noise, closer to the initial time, in particular if the observer gain is high. The sensitivity to noise is expected to reduce when the gain is reduced. In Section 3.1, we show that for systems with skew-adjoint generators, the initial state estimate given by the BFN method with colocated feedback, converges to the minimizer of the discrepancy between the measured and modeled outputs—given that the observer gains are taken to zero with a suitable rate. Here modeled output means the output generated by the dynamics equations without the unknown disturbances, when initialized from the estimated initial state. Systems with essentially skew-adjoint and dissipative (ESAD) generators, that is, $\mathcal{D}(A^*) = \mathcal{D}(A)$ and $A + A^* = -Q$ for some bounded $Q \geq 0$, are treated in Section 3.2. Then the colocated feedback has to be corrected by the operator $e^{At}e^{A^*t}$ in order to obtain such convergence (without this the BFN method converges to a biased estimate). In some special cases, this operator, or a feasible approximation for it, can be found analytically. In Section 4, we show that for the wave equation with constant dissipation, $u_{tt} = \Delta u - \epsilon u_t$ with Dirichlet boundary conditions, it holds that $e^{At}e^{A^*t} \approx e^{-\epsilon t}I$ resulting in a simple discounting factor for the observer gain. We shall also give upper bounds for the error due to the approximation $k(t)I \approx e^{At}e^{A^*t}$ in the observer gain. These bounds are given in the presented wave equation context, but their generalization would be straightforward.

E-mail address: atte.ej.aalto@gmail.com.

http://dx.doi.org/10.1016/j.sysconle.2016.06.002
0167-6911/

In the paper we use notation $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ for the space of bounded linear operators from a Hilbert space $\mathcal{H}_1$ to another Hilbert space $\mathcal{H}_2$. We also denote $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$. When there is no possibility of confusion, the notation $\|\cdot\|$ is used without indication in which space the norm is computed.

2. Problem setup and the back and forth nudging method

Consider the problem of retrieving the initial state of the system

$$\begin{cases} \dot{z} = Az + f + \eta, \\ z(0) = z_0, \\ y = Cz + \nu \end{cases} \tag{1}$$

from measurements $y(t)$ for $t \in [0, T]$. Here $A : \mathcal{D}(A) \to \mathcal{X}$ is the generator of a contraction semigroup e^{At} on the state space $\mathcal{X}$. The output operator $C : \mathcal{X} \to \mathcal{Y}$ is assumed to be bounded and both $\mathcal{X}$ and $\mathcal{Y}$ are assumed to be separable Hilbert spaces. The load term f is assumed to be known and η and ν are unknown input and output noise terms, respectively. Of the load and noise terms we assume $f, \eta \in H^1([0, T], \mathcal{X}_{-1})$ and $\nu \in L^2([0, T], \mathcal{Y})$ so that by [6, Theorem 4.1.6], z is a continuous $\mathcal{X}$-valued function and clearly then $y \in L^2([0, T], \mathcal{Y})$. Here $\mathcal{X}_{-1}$ is the extrapolation space obtained by completing $\mathcal{X}$ with respect to the norm $\|x\|_{\mathcal{X}_{-1}} = \left\|(\beta I - A)^{-1}x\right\|_{\mathcal{X}}$ where $\beta \in \rho(A)$ is fixed, see [6, Section 2.10].

The back and forth nudging method is defined as follows. The dynamics of the *forward observer* for $j = 1, 2, \dots$ are governed by

$$\begin{cases} \dot{z}_j^+(t) = Az_j^+(t) + f(t) + \kappa_j C^*(y(t) - Cz_j^+(t)), \\ z_j^+(0) = z_{j-1}^-(T), \quad \text{for } j \geq 2. \end{cases} \tag{2}$$

For $j = 1$, the initial state can be any vector in $\mathcal{X}$, since its influence on the initial state estimate will vanish as $j \to \infty$. The *backward observer* is also defined "forward in time"

$$\begin{cases} \dot{z}_j^-(t) = -Az_j^-(t) - f(T-t) + \kappa_j C^*(y(T-t) - Cz_j^-(t)), \\ z_j^-(0) = z_j^+(T), \quad \text{for } j \geq 1, \end{cases} \tag{3}$$

that is, $z_j^-(t)$ is an estimate of $z(T-t)$ and the initial state estimate that we are interested in is given by $z_j^-(T)$. The feedback of the form $C^*(y - Cz)$ in the observers is called *colocated feedback*, roughly meaning that the measurement through C and the control action through C^* take place in the same physical location in the computational domain. Classical references on the colocated feedback are [7] by Liu for skew-adjoint operators and bounded observation operators C, and [8] by Curtain and Weiss studying also ESAD operators and unbounded C. For a study on the colocated feedback for the wave equation, see [9] by Chapelle et al.

We show for systems with skew-adjoint generators, that if the observer gains κ_j in the back and forth nudging iterations (2) and (3) are selected in a certain way, then the initial state estimate will converge to the minimizer of the cost function

$$J(x) := \frac{1}{2} \int_0^T \|y(s) - Cz[x](s)\|^2 \, ds \tag{4}$$

where $z[x]$ is the solution of

$$\begin{cases} \dot{z}[x] = Az[x] + f, \\ z[x](0) = x. \end{cases} \tag{5}$$

Complementary results are obtained for systems with ESAD generators and for the classical BFN method with constant feedback $\kappa_j = \kappa$.

In the first results on the BFN method, [3,4], the feedback is simply given by a matrix K that can be chosen freely. Obviously it can be chosen so that both $A - K$ and $-A - K$ have strictly negative eigenvalues. Then if there are no noises, the BFN algorithm converges exponentially to the true initial state. The article [5] lays the foundation for the algorithm for infinite-dimensional systems. There the feedback in the observers is of the form $\pm A - K_\pm C$ where C is a given (possibly unbounded) observation operator and the feedback operator $K_\pm$ can be chosen freely. The main result itself is similar as that of [3,4], namely exponential convergence to the true initial state if $K_\pm$ can be chosen so that $\pm A - K_\pm C$ generate exponentially stable semigroups, and if the output is not corrupted by noise. Numerical aspects of the method are considered by Haine and Ramdani in [10]. The BFN method in the special case of systems with skew-adjoint operators with colocated feedback is studied by Ito et al. in [11] and by Phung and Zhang in [12]. In the latter article the method is called *time reversal focusing* and they treat the concrete problem of retrieving the initial state of the Kirchhoff plate equation from partial field measurements. Further development of the BFN method includes [13] by Haine showing a partial convergence result when the exact observability assumption is not satisfied, and [14] by Fridman extending the result to a class of semilinear systems. Application to unbounded computational domain is considered by Fliss et al. in [15], and a variant for systems containing a diffusive term is suggested by Auroux et al. in [16] where the idea is to change the sign of the diffusive term in the backward phase. The effect of input and output noise on the method has been briefly discussed by Shim et al. in [17] and by Donovan et al. in [18]. The BFN method or a related time-reversal approach can also be used for source identification problems, as in [19] by Ammari et al.

3. Results

We shall start by showing an important lemma. In the most general cases treated in this paper, we have feedbacks of the form $A - \kappa K(t)C^*C$ for the forward observer and $-A - \kappa K(T-t)C^*C$ for the backward observer where $K(\cdot) \in C_s([0, T], \mathcal{L}(\mathcal{X}))$, that is, $K(\cdot)$ is strongly continuous. We remark that when A is ESAD, then also $-A$ generates a strongly continuous semigroup since it can be regarded as a bounded perturbation of a skew-adjoint operator $A_0^* = -A - Q/2$ (see [20, Theorems II.3.24 and III.1.3]). For any $x \in \mathcal{X}$, it holds that

$$\frac{d}{dt}\left\|e^{-At}x\right\|^2 = \left\langle Qe^{-At}x, e^{-At}x\right\rangle \leq \|Q\| \left\|e^{-At}x\right\|^2$$

and so by Gronwall's inequality, $\left\|e^{-At}\right\| \leq e^{\|Q/2\|t}$.

Since also $K(t)C^*C$ is bounded and strongly continuous, then by [21, Theorem 3.9], the operators $A - \kappa K(t)C^*C$ and $-A - \kappa K(T-t)C^*C$ generate strongly continuous time evolution operators $U^+(t, s)$ and $U^-(t, s)$, respectively. We also refer to [20, VI.9c] for a background on perturbations of evolution families. Define also $U^\pm(t) := U^\pm(t, 0)$. As will be seen later in the proofs of our main results, after every forward and backward iteration, the old error term is multiplied by $U^-(T)U^+(T)$. We now show that if the dissipative term Q is small enough, and if $K(t) \approx k(t)I$ for some strictly positive function $k(\cdot)$, then this operator is strictly contractive.

Lemma 3.1. *Assume that the system (1) is exactly observable at time T, that is, $\int_0^T \left\|Ce^{At}x\right\|^2 dt \geq \delta \|x\|^2$ for all $x \in \mathcal{X}$ and some $\delta > 0$. Assume also $\mathcal{D}(A^*) = \mathcal{D}(A)$ and $A + A^* = -Q$ with bounded $Q \geq 0$, and that there exists a function $k \in C(0, T)$ with $k_1 \geq k(t) \geq k_0 > 0$, so that Q, $K(t)$, and $k(t)$ satisfy*

$$\alpha := 2k_0\delta - 2\|C\|^2 \left(2k_1\left(\frac{e^{\|Q/2\|T} - 1}{\|Q/2\|} - T\right) + \int_0^T e^{\|Q/2\|s} \|K(s) - k(s)I\| \, ds\right) > 0.$$

Then $\left\|U^-(T)U^+(T)\right\|_{\mathcal{L}(\mathcal{X})} \leq 1 - \alpha\kappa + \mathcal{O}(\kappa^2)$.