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Output-feedback synchronizability of linear time-invariant systems☆

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ABSTRACT

The paper studies the output-feedback synchronization problem for a network of identical, linear time-invariant systems. A criterion to test network synchronization is derived and the class of output-feedback synchronizable systems is introduced and characterized by sufficient and necessary conditions. In particular it is observed that output-feedback stabilizability is sufficient but not necessary for output-feedback synchronizability. In the special case of single-input single-output systems, conditions are derived in the frequency domain. The theory is illustrated with several examples.

1. Introduction

Synchronization has been recently a popular subject in the systems control community. This interest is motivated by the large array of phenomena exhibiting synchronization properties in physics and biology [1]. Moreover, distributed problems arising in engineering applications, are commonly addressed in the context of synchronization theory [2–5].

We consider N identical linear time-invariant (LTI) systems $\mathcal{P} = (A, B, C)$

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ y_i &= Cx_i, \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^q$, $i = 1, \ldots, N$, $N > 1$. The collection of systems (1) is denoted by $\mathcal{P}^N$. The systems are coupled according to the following feedback

$$u_i = K \sum_{j=1}^{N} \sigma_{i,j}(y_j - y_i), \quad i = 1, \ldots, N, \tag{2}$$

where $K \in \mathbb{R}^{m\times q}$. The problem of static output-feedback synchronization is to determine a matrix gain K and an interconnection topology, defined by the coefficients $\sigma_{i,j} \in \mathbb{R}$, such that the solutions of (1), (2) asymptotically synchronize, i.e. $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$ for every i, j and every initial conditions. Both existence and design questions are of interest. In this paper we will address the existence question: determine under what conditions on (1), a matrix K and a communication topology $\sigma_{i,j}$ exist such that the solutions of (1), (2) synchronize. We will call this property *static output-feedback synchronizability* or, for short, *synchronizability*. The design problem is subject of ongoing research.

The output-feedback synchronization problem has been addressed in [6] by assuming that B is the identity matrix and in [7] by assuming that C is the identity matrix. Both scenarios are particular cases of the general framework considered in this paper. In [8] the synchronization problem is addressed by assuming that the columns of B are contained in the image of C^T. Finally, a number of publications, see e.g., [9,10], study synchronization for specific systems such as double integrators and harmonic oscillators.

As for the output-feedback stabilization problem, the limitations imposed by static output-feedback can be overcome by using dynamic controllers. In [11] and [12] it has been shown that, assuming that the interconnection topology satisfies a minimal connectivity requirement, stabilizability and detectability of the isolated systems is sufficient for the existence of a dynamic controller synchronizing the network. In [11] the solution has been proposed in the case of time-varying communication topologies. Finally, [13] addressed the synchronization problem when the systems composing the network are not identical.

As shown in this paper, stabilizability and detectability are not sufficient for synchronizability. We first show that the synchronization problem can be addressed by studying the so called synchronization region (which depends on the structural properties of the uncoupled systems and the controller gain K) and the location of the eigenvalues of the interconnection matrix (which must be located inside the synchronization region in order for the network to

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synchronize). A connection between synchronizability and output-feedback stabilizability is established. It is shown that, somehow surprisingly, output-feedback stabilizability of the systems composing the network is a sufficient but not necessary condition for synchronizability. The notion of synchronization region and the synchronization criterion are then used to derive a graphical test to check synchronizability in networks of SISO systems.

The paper is organized as follows. Section 2 introduces the notation used throughout the paper and reviews preliminary material. Section 3 formalizes the synchronization problem. Sections 3 and 4 present the main results of the paper. We conclude the paper by illustrating the theory with some examples and with some final remarks. Preliminary results related to this paper appeared in [14].

2. Preliminaries

2.1. Notations

The following notations will be used throughout the paper. We denote the open right (left) half complex plane by $\mathbb{C}_{>0}$ ($\mathbb{C}_{<0}$), and the closed right (left) half complex plane by $\mathbb{C}_{\geq 0}$ ($\mathbb{C}_{\leq 0}$). We denote by $\mathbf{1}_n$ the column vector in $\mathbb{C}^n$ containing 1 in each entry. Given a complex matrix $M \in \mathbb{C}^{n\times m}$, M^T denotes its transpose and M^* its conjugate transpose. Given a square matrix $M \in \mathbb{C}^{n\times n}$, $\sigma\{M\}$ denotes its spectrum (defined as the multiset of the eigenvalues of M). The matrix M is called Hurwitz if $\sigma\{M\} \subseteq \mathbb{C}_{<0}$. We write $M > 0$ ($M \geq 0$) to indicate that M is positive-definite (positive-semidefinite). The identity matrix in $\mathbb{C}^{n\times n}$ is denoted by I_n.

2.2. Graph theory

A *directed graph* $\mathcal{G}$ consists of the triple $(\mathcal{V}, \mathcal{E}, \Sigma)$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of edges and $\Sigma \in \mathbb{R}^{N\times N}$ is a weighted adjacency matrix. Each element $\sigma_{i,j}$ (an element of Σ) is nonzero if and only if $(i,j) \in \mathcal{E}$. When $(i,j) \in \mathcal{E}$, node j is called a *neighbor* of node i. We assume that there are no self-loops and therefore $\sigma_{i,i} = 0$ for $i = 1 \dots N$. Unless differently stated, we allow for negative weights $\sigma_{i,j}$. The set of graphs with the properties above is denoted by $\mathscr{G}^N$. Two subsets of $\mathscr{G}^N$ are given special notations: $\mathscr{G}_+^N$ is the subset of graphs with non-negative weights ($\sigma_{i,j} \geq 0$); while $\mathscr{G}_u^N$ is the subset of graphs characterized by symmetric matrices Σ. Given a graph $\mathcal{G} \in \mathscr{G}_+^N$, a path between two nodes n_1, n_l is a sequence of nodes $\{n_1, n_2, \dots, n_l\}$ such that n_i, n_{i+1} is an edge for $i = 1, \dots, l-1$. A node n_b is called *reachable* from a node n_a if there exists a path between n_a and n_b. A node is *globally reachable* if it is reachable from every other node. Given a graph $\mathcal{G} \in \mathscr{G}^N$, we define the *interconnection matrix* $L_\mathcal{G}$ as the $N\times N$ matrix with elements

$$[L_\mathcal{G}]_{i,j} := \begin{cases} \sum_{k=1}^{N} \sigma_{i,k}, & i = j, \\ -\sigma_{i,j}, & i \neq j. \end{cases} \tag{3}$$

The matrix $L_\mathcal{G}$ always contains 0 and $\mathbf{1}_N$ as an eigenvalue–eigenvector pair (since $L_\mathcal{G}$ has zero row sum). $L_\mathcal{G}$ has special properties when the graph $\mathcal{G}$ belongs to $\mathscr{G}_+^N$ or $\mathscr{G}_u^N$. For graphs in $\mathscr{G}_u^N$, $L_\mathcal{G}$ is a symmetric matrix and has therefore real eigenvalues. For graphs in $\mathscr{G}_+^N$, the associated interconnection matrix $L_\mathcal{G}$ is called *Laplacian matrix*, and it is the generalization of the standard Laplacian matrix defined for undirected graphs (see e.g., [15] and references therein). All the eigenvalues of a Laplacian matrix have non-negative real part and the (always present) zero eigenvalue has multiplicity one if and only if the graph contains a globally reachable node [16]. Let Q_N be a matrix belonging to $\mathbb{R}^{(N-1)\times N}$ and satisfying the following properties

$$Q_N\mathbf{1}_N = 0, \qquad Q_N^TQ_N = \Pi_N, \qquad Q_NQ_N^T = I_{N-1}, \tag{4}$$

where $\Pi_N := I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$ is the projector onto the subspace orthogonal to span$(\mathbf{1}_N)$. Given a graph $\mathcal{G}$, the *reduced interconnection matrix* is defined by

$$\tilde{L}_\mathcal{G} := Q_N L_\mathcal{G} Q_N^T \tag{5}$$

where $Q_N \in \mathbb{R}^{(N-1)\times N}$ and satisfies the properties (4). The spectrum of $\tilde{L}_\mathcal{G}$ is the spectrum of $L_\mathcal{G}$ with one instance of the zero eigenvalue removed, i.e. $\sigma\{\tilde{L}_\mathcal{G}\} = \sigma\{L_\mathcal{G}\} \setminus \{0\}$ [17]. Therefore, when $\mathcal{G} \in \mathscr{G}_+^N$ contains a globally reachable node, $\sigma\{\tilde{L}_\mathcal{G}\} \subset \mathbb{C}_{>0}$. These properties are invariant to the choice of Q_N [17].

3. Synchronization criterion and synchronizability

We represent the network coupling structure with a directed graph. For this purpose we introduce N nodes labeled consecutively from 1 to N. Each node represents a system in the network. If a coefficient $\sigma_{i,j} = 0$ then the edge connecting node i to node j is not present. If $\sigma_{i,j} \neq 0$ the relative edge exists and its weight is determined by the (possibly negative) coefficient $\sigma_{i,j}$. We call the resulting graph $\mathcal{G}$ the *communication topology*. A collection of systems (1) together with a feedback matrix $K \in \mathbb{R}^{m\times q}$ and a communication topology $\mathcal{G}$ form a *network* that will be denoted by $\mathcal{N} := (\mathcal{P}^N, K, \mathcal{G})$. The next definition formalizes the notion of network synchronization.

Definition 1. A network $\mathcal{N} = (\mathcal{P}^N, K, \mathcal{G})$, is said to *synchronize* if

$$\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0,$$

for $i, j = 1, 2, \dots, N$ and for all initial conditions.

Network synchronization depends on the structural properties of the system $\mathcal{P}$, on the graph $\mathcal{G}$ and on the choice of the matrix $K \in \mathbb{R}^{m\times q}$. In this paper we investigate the structural properties of $\mathcal{P}$ such that $(\mathcal{P}^N, K, \mathcal{G})$ synchronizes for some K and $\mathcal{G}$.

Definition 2 (*Synchronizability*)**.** A collection of systems $\mathcal{P}^N$ is *output-feedback synchronizable* (OFS) if there exist a matrix $K \in \mathbb{R}^{m\times q}$ and a graph $\mathcal{G} \in \mathscr{G}^N$ such that the network $(\mathcal{P}^N, K, \mathcal{G})$ synchronizes.

We will make use of the following notion of *synchronization region*.

Definition 3. Given a system $\mathcal{P} = (A, B, C)$ and a matrix $K \in \mathbb{R}^{m\times q}$, the *synchronization region* $\mathcal{S}_\mathcal{P}(K)$ is the subset of the complex plane defined by

$$\mathcal{S}_\mathcal{P}(K) := \left\{ s \in \mathbb{C} \,\middle|\, A - sBKC \text{ is Hurwitz} \right\}. \tag{6}$$

The term synchronization region is justified by the synchronization criterion presented below.

Theorem 1. *A network $\mathcal{N} = (\mathcal{P}^N, K, \mathcal{G})$ synchronizes if and only if $\sigma\{\tilde{L}_\mathcal{G}\} \subseteq \mathcal{S}_\mathcal{P}(K)$.*

Proof. Define $x = [x_1^T, \dots, x_N^T]^T$ and rewrite (1), (2) in compact form as

$$\dot{x} = (I_N \otimes A - L_\mathcal{G} \otimes BKC)x.$$

Let $\mathcal{X}_\parallel := \{x \in \mathbb{R}^{nN} \mid (\Pi_N \otimes I_n)x = 0\}$ be the synchronization subspace and $\mathcal{X}_\perp := \{x \in \mathbb{R}^{nN} \mid \left(\frac{1}{N}1_N1_N^T \otimes I_n\right)x = 0\}$ its orthogonal complement, called the transversal subspace. The network synchronizes if and only if, for any initial conditions, the projection of