Systems & Control Letters 94 (2016) 165–171

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Optimal control of discrete-time bilinear systems with applications to switched linear stochastic systems[☆]

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ARTICLE INFO

Article history:
Received 7 January 2016
Received in revised form
28 April 2016
Accepted 6 June 2016
Available online 12 July 2016

Keywords:
Discrete-time
Stability analysis
Optimal control
Bilinear system
Switched linear stochastic system

ABSTRACT

This paper aims at characterizing the most destabilizing switching law for discrete-time switched systems governed by a set of bounded linear operators. The switched system is embedded in a special class of discrete-time bilinear control systems. This allows us to apply the variational approach to the bilinear control system associated with a Mayer-type optimal control problem, and a second-order necessary optimality condition is derived. Optimal equivalence between the bilinear system and the switched system is analyzed, which shows that any optimal control law can be equivalently expressed as a switching law. This specific switching law is most unstable for the switched system, and thus can be used to determine stability under arbitrary switching. Based on the second-order moment of the state, the proposed approach is applied to analyze uniform mean-square stability of discrete-time switched linear stochastic systems. Numerical simulations are presented to verify the usefulness of the theoretic results.

1. Introduction

The motivation for the study of switched systems has its roots in industry for two aspects in general. On one hand, many physical systems encountered in practice exhibit switching among a set of subsystems depending on various environmental factors; see e.g., [1] for more details. On the other hand, switched multi-controller systems have numerous applications in the control of mechanical systems, process control, automotive industry, and many other fields [2]. Analysis and synthesis of a switched system have attracted much attention from control community, and fruitful achievements have been developed; see [3–8], and the references therein.

A common problem for a switched linear system is that of determining whether it is stable under arbitrary switching. Much effort has been made to approach this problem, resulting in various methods and tools; see the recent survey paper [9,10] and the references therein. This problem is closely related to determine the *joint spectral radius* (JSR) of the set of subsystems. However, as shown in [11], computing the JSR is extremely hard. The difficulty mainly stems from the fact that there exist infinite switching laws. To circumvent this obstacle, a natural idea is characterizing the most destabilizing switching laws. This enables us to compute the JSR along several specific switching laws and analyze stability under arbitrary switching. Finding the most unstable switching law is an optimal control issue, it naturally reminds us of the variational approach [12]. Recently, considerable research efforts have been directed towards the development of variational approach in the stability analysis of switched systems [13–15]. However, it is noticed the aforementioned schemes are only applicable to deterministic switched systems. Extra efforts are needed to analyze stability under arbitrary switching in a stochastic setting.

In real world, not all technological processes can be adequately represented by deterministic systems such as chemical process and biology engineering [16,17]. Besides, to handle ubiquitous uncertainties in realistic system models, one can also describe uncertainties using stochastic models and design a control strategy to meet the design criteria. Researchers are directed to approach switched stochastic systems from various directions [16–21]. Many nice works in stochastic stability of jump linear systems have been reported; see [22,23], and the references therein. Based on dwell time or average dwell time constraints, mean-square stability of switched linear stochastic systems has also been

☆ This work is supported by National Natural Science Foundation of China under Grant 61403018 and 61473024.
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http://dx.doi.org/10.1016/j.sysconle.2016.06.004
0167-6911/

studied in [16,19,24]. In spite of progress, it should be mentioned that these results concerning stability analysis impose restrictions on the switching signal. To the best of our knowledge, there are limited results dealing with stability analysis of switched linear stochastic systems under arbitrary switching. The difficulty lies in determining the JSR of a set of bounded linear operators.

This paper attempts to characterize the "most unstable" switching law for discrete-time switched systems (DSS) governed by a set of bounded linear operators. To apply the variational approach pioneered by E.S. Pyatnitskii, we embed the DSS in a generalized class of discrete-time bilinear systems (DBS). A second-order necessary optimality condition is derived for the DBS associated with a Mayer-type optimal problem. Optimal equivalence between the DBS and the DSS is then analyzed, which indicates that any optimal control can be equivalently expressed as a switching law. This specific switching law is most unstable for the DSS, and thus can be used to compute the JSR. Based on the second-order moment of the state, the proposed approach is applied to analyze global uniform mean-square stability (GUMS) of discrete-time switched linear stochastic systems (DSLSS).

The rest of the paper is organized as follows. The problem is formulated in Section 2. Section 3 presents the main results, followed by the application to GUMS analysis of the DSLSS in Section 4. Numerical simulations are presented in Section 5 and conclusions are made in the final section.

Notations: $\mathcal{S}_n$ is the Hilbert space composed of $n \times n$ symmetric matrices with the inner product $\langle \cdot, \cdot \rangle$ defined by $\langle \boldsymbol{Y}_1, \boldsymbol{Y}_2 \rangle = \mathrm{tr}(\boldsymbol{Y}_1 \boldsymbol{Y}_2)$, $\forall \boldsymbol{Y}_1, \boldsymbol{Y}_2 \in \mathcal{S}_n$. The set of natural numbers and n-dimensional real vectors are respectively denoted by $\mathbb{N}$ and $\mathbb{R}^n$. Denote by $\mathscr{L}(\mathcal{S}_n)$ the set of all bounded linear operators from $\mathcal{S}_n$ to $\mathcal{S}_n$. For $\mathcal{L} \in \mathscr{L}(\mathcal{S}_n)$, let $\|\mathcal{L}\| = \max_{\|\boldsymbol{Y}\|=1, \boldsymbol{Y} \in \mathcal{S}_n} \|\mathcal{L}(\boldsymbol{Y})\|$. Throughout the paper, denote by $\|\cdot\|$ the norm of a vector in $\mathbb{R}^n$, a matrix in $\mathcal{S}_n$ or a operator in $\mathscr{L}(\mathcal{S}_n)$ induced by $\langle \cdot, \cdot \rangle$ without ambiguity. $\mathbb{E}$ stands for the mathematical expectation with respect to the given probability measure $\mathscr{P}$, and $\mathrm{tr}(\cdot)$ denotes the trace of a square matrix. $\boldsymbol{e}^i_{m-1}$ denotes the ith column of the $(m-1) \times (m-1)$ identity matrix. The symbol $\mathrm{vec}(\cdot)$ represents the linear operator stacking the entries of a matrix columnwise, and while $\mathrm{vec}^{-1}(\cdot)$ denotes the linear inverse operator. $\sigma_{\max}(\cdot)$ denotes the maximal eigenvalue of a positive semi-definite matrix. For a vector or a matrix $\boldsymbol{Y}$, we write $\boldsymbol{Y} \prec 0 (\succeq 0)$ if all elements of $\boldsymbol{Y}$ are (no) less than 0.

2. Problem formulation

Let $\mathbb{L} = \{\mathcal{L}_i \in \mathscr{L}(\mathcal{S}_n)\}_{i \in \mathcal{M}}$ with $\mathcal{M} = \{0, 1 \ldots, m-1\}$. Consider the following DSS described by $\mathbb{L}$

$$\mathbf{Z}(k+1) = \mathcal{L}_{\gamma(k)}(\mathbf{Z}(k)), \qquad \mathbf{Z}(0) = \mathbf{Z}_0, \quad k \in \mathbb{N} \tag{1}$$

where $\mathbf{Z}(k) \in \mathcal{S}_n$ is state, and $\gamma(\cdot) : \mathbb{N} \to \mathcal{M}$ is the switching law. This models a system that can switch among the m linear subsystems $\mathbf{Z}(k+1) = \mathcal{L}_i(\mathbf{Z}(k))$, $i \in \mathcal{M}$ with the switching law determining which system is active at each time step. DSS (1) represents various discrete-time switched linear systems in both deterministic and stochastic settings, see [25,26].

Definition 1. DSS (1) is said to be globally uniformly asymptotically stable (GUAS) if for any $\mathbf{Z}_0 \in \mathcal{S}_n$ and any switching law $\gamma(k)$, $\lim_{k\to\infty} \|\mathbf{Z}(k)\|^2 = 0$.

Definition 2. For an arbitrarily fixed final time $N \in \mathbb{N}$, a switching law maximizing $\|\mathbf{Z}(N)\|^2$ is referred as the "most unstable" switching law and denoted by $\gamma^*(k)$.

Problem 1. For an arbitrarily fixed final time $N \in \mathbb{N}$, find the "most unstable" switching law $\gamma^*(k)$.

As shown in [27], DSS (1) is GUAS if and only if the JSR of $\mathbb{L}$ satisfies $\rho(\mathbb{L}) < 1$, where $\rho(\mathbb{L}) = \lim_{k\to\infty} \hat{\rho}_k(\mathbb{L})$ with $\hat{\rho}_k(\mathbb{L}) = \max\{\|\mathcal{L}_{i_k} \cdots \mathcal{L}_{i_0}\|^{\frac{1}{k+1}} : i_j \in \mathcal{M}\}$. A natural idea is to characterize $\gamma^*(k)$, and analyze the corresponding trajectory $\mathbf{Z}^*(k)$. It is clear that if $\|\mathbf{Z}^*(k)\|^2$ converges to origin along with time evolution, so does the norm of any other solution. Via this transformation, we only need to compute the operator norm along with $\gamma^*(k)$ for determining the JSR. To apply the variational approach, we embed DSS (1) in the following DBS described by $\mathbb{L}$

$$\mathbf{Z}(k+1) = \left(\mathcal{L}_0 + \sum_{i=1}^{m-1} u_i(k)\mathcal{B}_i\right)(\mathbf{Z}(k)), \quad \boldsymbol{u}(k) \in \mathcal{U} \tag{2}$$

where $\mathcal{B}_i = \mathcal{L}_i - \mathcal{L}_0$, the control set $\mathcal{U}$ is given by

$$\mathcal{U} = \left\{\boldsymbol{u}(k) \in \mathbb{R}^{m-1} : u_i(k) \geq 0, \sum_{i=1}^{m-1} u_i(k) \leq 1\right\} \tag{3}$$

with $\boldsymbol{u}(k) = (u_1(k) \ \ldots \ u_{m-1}(k))' \in \mathcal{U}$. When $\boldsymbol{u}(k)$ is bang–bang, taking values in the set $\{\mathbf{0}, \boldsymbol{e}^1_{m-1}, \ldots, \boldsymbol{e}^{m-1}_{m-1}\}$ for any $k \in \mathbb{N}$, DBS (2) reduces to DSS (1). Fix an arbitrary final time $N \in \mathbb{N}$, and consider the Mayer-type optimal problem

$$\max_{\boldsymbol{u}(k)\in\mathcal{U}} J(N; \boldsymbol{u}(k), \mathbf{Z}_0) = \max_{\boldsymbol{u}(k)\in\mathcal{U}} \|\mathbf{Z}(N; \boldsymbol{u}(k), \mathbf{Z}_0)\|^2, \tag{4}$$

where $\mathbf{Z}(N; \boldsymbol{u}(k), \mathbf{Z}_0)$ denotes the solution of (2) corresponding to $\boldsymbol{u}(k)$ at time N. We refer such a control as an *optimal control*, denoted by $\boldsymbol{u}^*(k)$. If there always exists an optimal control $\boldsymbol{u}^*(k)$ that is bang–bang, then $\gamma^*(k)$ for DSS (1) and $\boldsymbol{u}^*(k)$ for DBS (2) are equivalent with respect to the issue of maximizing $\|\mathbf{Z}(N)\|^2$.

Definition 3. We say DBS (2) is globally asymptotically stable (GAS) if for any $\boldsymbol{u}(k) \in \mathcal{U}$, $\mathbf{Z}_0 \in \mathcal{S}_n$, $\lim_{N\to\infty} \mathbf{Z}(N; \boldsymbol{u}(k), \mathbf{Z}_0) = \mathbf{0}$.

Definition 4. For any $\boldsymbol{u}(k) \in \mathcal{U}$, a perturbation $\boldsymbol{v}(k)$ is said to be admissible if $\boldsymbol{u}(k) + \boldsymbol{v}(k) \in \mathcal{U}$.

Lemma 1. *Let $\boldsymbol{w} = [w_1 \ \ldots \ w_{m-1}]' \in \mathbb{R}^{m-1}$ and $\boldsymbol{G} = (G_{ij}) \succeq \mathbf{0} \in \mathcal{S}_{m-1}$ be given. For any $\boldsymbol{u} \in \mathcal{U}$, define $f(\boldsymbol{v}) = \boldsymbol{v}'\boldsymbol{w} + \boldsymbol{v}'\boldsymbol{G}\boldsymbol{v}$, where $\boldsymbol{v} \in \mathbb{R}^{m-1}$ is any admissible perturbation of $\boldsymbol{u}$. Suppose that $f(\boldsymbol{v}) \leq 0$ for all admissible perturbations, then the following statements hold:* (i) *If $\boldsymbol{w} \prec 0$, then $\boldsymbol{u} = 0$;* (ii) *If $w_i > 0$ and $w_i > w_j$ for any $j \neq i$, then $\boldsymbol{u} = \boldsymbol{e}^i_{m-1}$;* (iii) *If $\boldsymbol{w} = \mathbf{0}$, then $\boldsymbol{G} = \mathbf{0}$.*

Proof. Firstly, we prove (i) by contradiction. If $\boldsymbol{u} \neq \mathbf{0}$, it implies that there must exist an index j such that $u_j > 0$. Choose a sufficiently small positive scalar ϵ such that $\boldsymbol{v} = -\epsilon \boldsymbol{e}^j_{m-1}$ is an admissible perturbation, we have $f(\boldsymbol{v}) = -\epsilon w_j + \epsilon^2 G_{jj} > 0$. This contradicts $f(\boldsymbol{v}) \leq 0$. Thus, we have $\boldsymbol{u} = \mathbf{0}$.

Secondly, assume that $w_i > 0$ and $w_i > w_j$ for any $j \neq i$, and $\boldsymbol{u} \neq \boldsymbol{e}^i_{m-1}$. Since $\boldsymbol{u} \neq \boldsymbol{e}^i_{m-1}$, there are two cases: $\boldsymbol{u} = \delta \boldsymbol{e}^i_{m-1}$ with $\delta \in [0, 1)$ and there exists an index j such that $u_j > 0$. For the former, choosing an admissible perturbation $\boldsymbol{v} = \epsilon \boldsymbol{e}^i_{m-1}$ yields $f(\boldsymbol{v}) > 0$; while for the latter, letting $\boldsymbol{v} = \epsilon(\boldsymbol{e}^i_{m-1} - \boldsymbol{e}^j_{m-1})$ be an admissible perturbation, we obtain $f(\boldsymbol{v}) = \epsilon(w_i - w_j) + \boldsymbol{v}'\boldsymbol{G}\boldsymbol{v} > 0$. Based on the above analysis, we have $\boldsymbol{u} = \boldsymbol{e}^i_{m-1}$.

Thirdly, for any $\boldsymbol{u} \in \mathcal{U}$, we can find a group of admissible perturbations $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{m-1}\}$ that is a basis of $\mathbb{R}^{m-1}$. Thus, the statement (iii) holds by noting $f(\boldsymbol{v}) \leq 0$ for all admissible perturbations. □

Definition 5. For a given $\mathcal{L} \in \mathscr{L}(\mathcal{S}_n)$, $\overline{\mathcal{L}}$ is said to be the Hilbert adjoint operator of $\mathcal{L}$ if it holds

$$\langle \mathcal{L}(\boldsymbol{Y}_1), \boldsymbol{Y}_2 \rangle = \langle \boldsymbol{Y}_1, \overline{\mathcal{L}}(\boldsymbol{Y}_2) \rangle, \quad \forall \boldsymbol{Y}_1, \boldsymbol{Y}_2 \in \mathcal{S}_n. \tag{5}$$

Lemma 2. *For any $\alpha \in \mathbb{R}$, $\mathcal{L}_1, \mathcal{L}_2 \in \mathscr{L}(\mathcal{S}_n)$, we have $\overline{\mathcal{L}_1\mathcal{L}_2} = \overline{\mathcal{L}_2}\ \overline{\mathcal{L}_1}$, $\overline{\overline{\mathcal{L}_1}} = \mathcal{L}_1$, $\overline{\alpha \mathcal{L}_1} = \alpha \overline{\mathcal{L}_1}$, $\|\overline{\mathcal{L}_1}\| = \|\mathcal{L}_1\|$.*