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Stability of the Kalman filter for continuous time output error systems

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ABSTRACT

The stability of the Kalman filter is usually ensured by the uniform complete controllability *regarding the process noise* and the uniform complete observability of linear time varying systems. This paper studies the case of continuous time *output error* systems, in which the process noise is totally absent. The classical stability analysis assuming the controllability regarding the process noise is thus not applicable. It is shown in this paper that the uniform complete observability *alone* is sufficient to ensure the asymptotic stability of the Kalman filter applied to time varying *output error* systems, regardless of the stability of the considered systems themselves. The exponential or polynomial convergence of the Kalman filter is then further analyzed for particular cases of stable or unstable output error systems.

1. Introduction

The well known Kalman filter has been extensively studied and is being applied in many different fields [1–5]. The purpose of the present paper is to study the stability of the Kalman filter in a particular case rarely covered in the literature: the absence of process noise in the state equation of a linear time varying (LTV) system. Such systems are known as *output error* (OE) systems. Though typically process noise and output noise are both considered in Kalman filter applications, the case with *no* process noise is of particular interest when state equations originate from physical laws that are believed accurate enough. Another motivation of this study is OE system identification [6–8]. When the prediction error method [9] is applied to OE system identification, the stability of the output predictor is a crucial issue. The results presented in this paper ensure the stability of the predictor based on the Kalman filter.

While the optimal properties of the Kalman filter are frequently recalled, its stability properties are less often mentioned in the recent literature. The classical stability analysis is based on both the uniform complete controllability *regarding the process noise* and the uniform complete observability of LTV systems [10,2]. In the case of OE systems, there is no process noise at all in state equations, hence the controllability regarding the process noise cannot be fulfilled, and the classical stability results are not applicable. The present paper aims at completing this missing case.

The optimal state estimation realized by the Kalman filter can be viewed as a trade-off between the uncertainties in the state equation and in the output equation. In an OE system, the state equation is assumed noise-free. This point of view suggests that the state estimation should solely rely on the state equation, provided that the initial state of the OE system is *exactly* known. In practice the Kalman filter remains useful when the initial state is not exactly known or when the OE system is unstable. Of course, if the state of an unstable system diverges, so does its state estimate by the Kalman filter. Typically in practice, unstable systems are stabilized by feedback controllers so that the system state remains bounded. The Kalman filter can be applied either to the controlled system itself or to the entire closed loop system. In the latter case, the controller must be linear and completely known, excluding the saturation protection and any other nonlinearities.

The classical *optimality* results of the Kalman filter are also valid in the case of OE systems [2, chapter 7]. However, it is necessary to complete the stability analysis, as the classical results are not applicable in this case.

The main results presented in this paper are as follows. Under the uniform complete observability condition, it is first shown that the dynamics of the Kalman filter applied to a continuous time OE LTV system is asymptotically stable, *regardless of the stability of the system itself.* The exponential or polynomial convergence of the Kalman filter is further established, depending on the stability or the instability property of the considered system. The boundedness of the solution of the Riccati equation, and thus also of the Kalman gain, is also proved under the same condition. These results complete the classical results [10,2], which exclude the case of OE systems.

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For linear time invariant (LTI) systems, it is a common practice to design the Kalman filter by solving an algebraic Riccati equation (in contrast to dynamic differential Riccati equation for general LTV systems as considered in the present paper). In this case, the controllability and observability conditions can be replaced by the weaker stabilizability and detectability conditions [11,12].

Some preliminary results about the Riccati equation of time invariant systems have been presented in the conference paper [13], and some further results on the asymptotic stability of the Kalman filter have been presented in the conference paper [14], without the exponential or polynomial convergence rate analysis and part of the numerical examples reported in the present paper.

The rest of the paper is organized as follows. Some preliminary elements are introduced in Section 2. The problem considered in this paper is formulated in Section 3. The asymptotic stability of the Kalman filter for OE systems is established in Section 4. Exponential and polynomial convergence rates of the Kalman filter are analyzed in Section 5. Some analytical and numerical examples are presented in Sections 6 and 7. Finally, concluding remarks are drawn in Section 8.

2. Definitions and preliminary elements

Let us shortly recall some definitions and basic facts about LTV systems, which are necessary for the following sections.

Let m and n be any two positive integers. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm. For a matrix $A \in \mathbb{R}^{m\times n}$, $\|A\|$ denotes the matrix norm induced by the Euclidean vector norm, which is equal to the largest singular value of A and known as the spectral norm when $m = n$. Then $\|Ax\| \leq \|A\|\,\|x\|$ for all $A \in \mathbb{R}^{m\times n}$ and all $x \in \mathbb{R}^n$. For two real square symmetric positive definite matrices A and B, $A > B$ means $A - B$ is positive definite.

Let $A(t) \in \mathbb{R}^{m\times n}$ be defined for $t \in \mathbb{R}$. It is said (upper) bounded if $\|A(t)\|$ is bounded.

Consider the homogeneous LTV system

$$\frac{dx(t)}{dt} = A(t)x(t) \tag{1}$$

with $x(t) \in \mathbb{R}^n$ and $A(t) \in \mathbb{R}^{n\times n}$, and let $\Phi(t, t_0)$ be the associated state transition matrix such that, for all $t, t_0 \in \mathbb{R}$, $d\Phi(t, t_0)/dt = A(t)\Phi(t, t_0)$ and $\Phi(t, t) = I_n$ with I_n denoting the $n \times n$ identity matrix.

Definition 1. System (1) is *Lyapunov stable* if there exists a positive constant γ such that, for all $t, t_0 \in \mathbb{R}$ satisfying $t \geq t_0$, the following inequality holds

$$\|\Phi(t, t_0)\| \leq \gamma. \quad \Box \tag{2}$$

This definition concerns the boundedness of the state vector, whereas the following definition ensures its convergence to zero.

Definition 2. System (1) is *asymptotically stable* if it is Lyapunov stable and if the following limiting behavior holds

$$\lim_{t\to+\infty} \|x(t)\| = 0 \tag{3}$$

for any initial state $x(t_0) \in \mathbb{R}^n$. $\Box$

To characterize the convergence rate of some stable systems, the exponential stability is defined as follows.

Definition 3. System (1) is *exponentially stable* if there exist two positive constants α, β such that the inequality

$$\|\Phi(t, t_0)\| \leq \beta e^{-\alpha(t-t_0)} \tag{4}$$

holds for all $t, t_0 \in \mathbb{R}$ satisfying $t \geq t_0$. $\Box$

The definitions of Lyapunov and exponential stabilities in Definitions 1 and 3 are based on the norm of the state transition matrix $\Phi(t, t_0)$. They could take another equivalent form, as suggested by the following lemma. See also [15].

Lemma 1. *Let $M \in \mathbb{R}^{n\times n}$ and ξ be some positive scalar value. Then $\|M\| \leq \xi$ if and only if $\|Mv\| \leq \xi\|v\|$ for all $v \in \mathbb{R}^n$.* $\Box$

See the Appendix at the end of this paper for a proof of the lemma.

With $M = \Phi(t, t_0)$, $\xi = \gamma$ or $\xi = \beta e^{-\alpha(t-t_0)}$, Lemma 1 suggests that Definitions 1 and 3 could be based on the norm of $x(t) = \Phi(t, t_0)x(t_0)$ instead of the norm of $\Phi(t, t_0)$.

Definition 4. System (1) is *strongly unstable* if there exist two positive constants α, β such that the inequality

$$\|x(t)\| \geq \beta e^{\alpha(t-t_0)}\|x(t_0)\| \tag{5}$$

holds for all $t, t_0 \in \mathbb{R}$ satisfying $t \geq t_0$ and for all $x(t_0) \in \mathbb{R}^n$. $\Box$

The following observability definition for LTV systems follows Kalman [10].

Definition 5. The matrix pair $[A(t), C(t)]$ with $A(t) \in \mathbb{R}^{n\times n}$ and $C(t) \in \mathbb{R}^{m\times n}$ is *uniformly completely observable* if there exist positive constants τ, ρ_1 and ρ_2 such that, for all $t \in \mathbb{R}$, the following inequalities hold[2]

$$\rho_1 I_n \leq \int_{t-\tau}^{t} \Phi^T(s, t)C^T(s)R^{-1}(s)C(s)\Phi(s, t)ds \tag{6}$$

$$\leq \rho_2 I_n \tag{7}$$

with some bounded symmetric positive definite matrix $R(s) \in \mathbb{R}^{m\times m}$ (typically the covariance matrix of the output noise in a stochastic state space system). $\Box$

The fact that observability is preserved by output feedback is stated in the following lemma.

Lemma 2. *Let $A(t) \in \mathbb{R}^{n\times n}$, $C(t) \in \mathbb{R}^{m\times n}$, $K(t) \in \mathbb{R}^{n\times m}$ be bounded piecewise continuous matrices. The matrix pair $[A(t), C(t)]$ is uniformly completely observable, if and only if the matrix pair $[A(t) - K(t)C(t), C(t)]$ is uniformly completely observable.* $\Box$

This lemma can be found in [16, chapter 2]. For more rigorous proofs see [17, chapter 2], [18, chapter 4], and [19].

3. Problem formulation and assumptions

In this section, the Kalman filter and its usual assumptions are first recalled, before the presentation of the particular case of OE systems considered in this paper.

3.1. Kalman filter and usual assumptions

The Kalman filter in continuous-time is *usually* designed for LTV systems modeled by

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + Q^{\frac{1}{2}}(t)d\omega(t) \tag{8a}$$

$$dy(t) = C(t)x(t)dt + R^{\frac{1}{2}}(t)d\eta(t) \tag{8b}$$

where $t \in \mathbb{R}$ represents the time, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^l$ the bounded input, $y(t) \in \mathbb{R}^m$ the output, $\omega(t) \in \mathbb{R}^n$, $\eta(t) \in \mathbb{R}^m$ are two independent Brownian processes with identity covariance matrices, $A(t), B(t), C(t), Q(t), R(t)$ are bounded

[2] Some variants of the definition of the uniform complete observability exist in the literature. The definition recalled here follows Kalman [10].