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Optimal control problem of backward stochastic differential delay equation under partial information[☆]

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ABSTRACT

In this paper, we study the optimal control problem of backward stochastic differential delay equation under partial information. A class of time-advanced stochastic differential equations (ASDEs) is introduced as the adjoint process via duality relation. By means of ASDEs, we suggest a maximum principle of this problem and obtain necessary and sufficient conditions of optimality. We apply the theoretical results to study linear quadratic optimal control problem and obtain a forward–backward stochastic differential filtering equation (FBSDFE) with time-advanced forward equation that leads to optimal control. At the same time, some auxiliary filtering results and properties about time-advanced SDEs are also presented.

1. Introduction

Backward stochastic differential equations (BSDEs) have been introduced in Pardoux and Peng [1]. Since then, the theory of BSDEs has been studied systematically. Research on related fields and their applications has become a notable endeavor among researchers in mathematical finance, optimal control, stochastic games and partial differential equations. A BSDE can be seen as an Itô stochastic differential equation (SDE) in which the terminal other than the initial condition is given. Under some conditions, BSDEs admit a pair of adapted solutions. Due to the above features, BSDEs are essentially different to forward SDEs. The corresponding stochastic control problems, whose systems are driven by BSDEs also have been widely studied by many authors, see [2–4].

Recently, a new class of backward stochastic differential equations with time-delayed generators was introduced by [5] and some novel applications of time-delayed BSDEs to problems related to pricing, hedging and investment portfolio management were provided in [6]. In contrast with standard BSDEs, time-delayed BSDEs involve their present value as well as their previous information. Due to the interesting structure and wide-range of applications, it is very necessary to study the dynamic optimizations of this type of equations. Recently, [7] obtained a maximum principle for one kind of time-delayed BSDEs. In their paper, they introduced a new class of stochastic differential equations called time-advanced SDEs. They also explored the duality relation between time-delayed BSDEs and time-advanced SDEs.

In many practical systems, the controller only gets partial information, instead of full information, such as an incomplete market. So, it is natural to study stochastic control problems of BSDEs under partial information. There have been some pioneering works

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in this direction. We refer readers to [8–10]. Our work distinguishes itself from the above ones in the following aspects. First, we study the more general cases: the backward systems are time-delayed and the information available to the controller is partial information. Because the adjoint equation is a time-advanced SDE, some necessary results about solution of this kind of SDE are presented. Second, we study linear quadratic optimal control problem systematically and obtain optimal control. Third, the forward–backward stochastic differential filtering equation, which was first introduced by [9], becomes a general FBSDE, i.e., forward SDE is time-advanced and BSDE is time-delayed. Some necessary filtering results about this kind of general FBSDE are given simultaneously. To the best of our knowledge, this kind of results has not been found in existing works.

The rest of this paper is organized as follows. In Section 2, we give some necessary notations and state some preliminary results about time-delayed BSDEs. In Section 3, we state the optimal control problem and establish necessary condition in the form of stochastic maximum principle for the optimal control. Also, we establish the corresponding sufficient condition. In Section 4, we apply the results obtained in Section 3 to study linear quadratic optimal control problem and obtain the corresponding FBSDFE, and then, we deduce the optimal control. In some special cases, we give the optimal control in the form of filtering feedback regulator. Some auxiliary filtering results are also presented in this section. Finally, we end this paper with a concluding remark.

2. Preliminary results

Throughout this paper, we denote by $\mathbb{R}^k$ the k-dimensional Euclidean space, $\mathbb{R}^{k\times l}$ the collection of $k\times l$ matrices. For a given Euclidean space, we denote by $\langle\cdot,\cdot\rangle$ (resp. $|\cdot|$) the scalar product (resp. norm). The superscript τ denotes the transpose of vectors or matrices. $\mathbb{E}^{\mathcal{F}_t}[\cdot]=\mathbb{E}[\cdot|\mathcal{F}_t]$ denotes the conditional expectation under filtration $\mathcal{F}_t$.

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$ be a complete filtered probability space equipped with an $m+\bar m$-dimensional, $\mathcal{F}_t$-adapted standard Brownian motion $(W(t),\bar W(t))$, where $\mathcal{F}=\mathcal{F}_T$. Let $T>0$ be the finite time duration and $0<\delta\leq T$ be the constant time delay. Moreover, we denote by $L^2(r,s;\mathbb{R}^k)$ the space of $\mathbb{R}^k$-valued deterministic functions $\varphi(t)$ satisfying $\int_r^s|\varphi(t)|^2dt<+\infty$, by $L^2(\mathcal{F}_t;\mathbb{R}^k)$ the space of $\mathbb{R}^k$-valued $\mathcal{F}_t$-measurable random variables ζ satisfying $\mathbb{E}|\zeta|^2<+\infty$, and by $L^2_{\mathcal{F}}(r,s;\mathbb{R}^k)$ the space of $\mathbb{R}^k$-valued $\mathcal{F}_t$-adapted processes $\psi(\cdot)$ satisfying $\mathbb{E}\int_r^s|\psi(t)|^2dt<+\infty$.

Let us now begin to consider a class of stochastic recursive delayed control problems described by the following time-delayed BSDE:

$$\begin{cases}-dy(t)=f(t,y(t),y(t-\delta),z(t),\bar z(t),v(t))dt-z(t)dW(t)\\ \quad-\bar z(t)d\bar W(t),\quad t\in[0,T],\\ y(T)=\xi,\qquad y(t)=\varphi(t),\quad t\in[-\delta,0).\end{cases}\tag{1}$$

Here $f:\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n\times m}\times\mathbb{R}^{n\times\bar m}\times\mathbb{R}^k\to\mathbb{R}^n$ is given continuous function in $(t,y,y_\delta,z,\bar z,v)$, $\xi\in L^2(\mathcal{F}_T;\mathbb{R}^n)$, $\varphi(t)$ is a deterministic function. Note that BSDE (1) is time-delayed, i.e., the BSDE's behavior at time t not only depends on the current situation, but also on a finite part of its past history. Such as the evolution of assets and other stochastic dynamical systems. To study the partial information optimal problem, we give some assumptions on v and f.

Let U be a nonempty convex subset of $\mathbb{R}^k$, and $\mathcal{G}_t\subseteq\mathcal{F}_t$ a given sub-filtration which represents the information available to the controller. Define the set of U_{ad} of admissible control to be the class of measurable processes satisfying the following conditions:

$$\begin{cases}\text{(i) } v(\cdot)\text{ is adapted to }\mathcal{G}_t\text{ and }v(t)\in U,\ t\in[0,T],\\ \text{(ii) }\mathbb{E}\displaystyle\int_0^T|v(t)|^2dt<\infty.\end{cases}$$

We also introduce the following assumption:

(H2.1) f is continuously differentiable in $(y,y_\delta,z,\bar z,v)$. Moreover, the partial derivatives $f_y,f_\delta,f_z,f_{\bar z},f_v$ of f with respect to $(y,y_\delta,z,\bar z,v)$ are uniformly bounded.

Then we have the following existence and uniqueness result which can be found in [7].

Lemma 2.1. *If $v(\cdot)$ is an admissible control and assumption* (H2.1) *holds, the time-delayed BSDE (1) admits a unique solution $(y(\cdot),z(\cdot),\bar z(\cdot))\in L^2_{\mathcal{F}}(-\delta,T;\mathbb{R}^n)\times L^2_{\mathcal{F}}(-\delta,T;\mathbb{R}^{n\times m})\times L^2_{\mathcal{F}}(-\delta,T;\mathbb{R}^{n\times\bar m})$ for sufficiently small $0\leq\delta\leq T$.*

The objective is to maximize the following functional over U_{ad}:

$$J(v(\cdot))=\mathbb{E}\int_0^T l(t,y(t),z(t),\bar z(t),v(t))dt+h(y(0)),$$

where l and h satisfy the following condition.

(H2.2) l and h are differentiable with respect to (y,z,v) and y respectively satisfying the condition that for each $v(\cdot)\in U_{ad}$, $l(\cdot,y,z,\bar z,v)\in L^1_{\mathcal{F}}(0,T;\mathbb{R})$, and all the derivatives are bounded.

In what follows, we will establish maximum principle of the above control problem.

3. Maximum principle

In this section, we derive a maximum principle of the optimal problem and obtain necessary and sufficient conditions of optimality.

Suppose $u(\cdot)$ is an optimal control of the problem. Then for any $0\leq\rho\leq 1$ and $v(\cdot)\in U_{ad}$, we take the variational control $v^\rho(\cdot)=u(\cdot)+\rho v(\cdot)$. Because U is convex, $v^\rho(\cdot)$ is in U_{ad}. For simplicity, we denote by $(y^{v^\rho}(\cdot),z^{v^\rho}(\cdot),\bar z^{v^\rho}(\cdot))$ and $(y^u(\cdot),z^u(\cdot),\bar z^u(\cdot))$ the corresponding state trajectories of system (1) with $v^\rho(\cdot)$ and $u(\cdot)$.

The following lemma that gives an estimation of $(y^u(\cdot),z^u(\cdot),\bar z^u(\cdot))$.

Lemma 3.1. *Let* (H2.1) *hold. Then we have*

$$\sup_{0\leq t\leq T}\mathbb{E}|y^{v^\rho}(t)-y^u(t)|^2\leq C\rho^2$$

$$\mathbb{E}\int_0^T|z^{v^\rho}(t)-z^u(t)|^2dt\leq C\rho^2$$

$$\mathbb{E}\int_0^T|\bar z^{v^\rho}(t)-\bar z^u(t)|^2dt\leq C\rho^2.$$

Proof. Using Itô's formula to $|y^{v^\rho}(t)-y^u(t)|^2$ and Gronwall's inequality, we draw the conclusion. □

Now we introduce the variational equation as follows:

$$\begin{cases}-dy_1(t)=[f^u_y(t)y_1(t)+f^u_{y_\delta}(t)y_1(t-\delta)+f^u_z(t)z_1(t)\\ \quad+f^u_{\bar z}(t)\bar z_1(t)+f^u_v(t)v(t)]dt-z_1dW(t)-\bar z_1d\bar W(t),\\ y_1(T)=0,\qquad y_1(t)=0,\quad t\in[-\delta,0),\end{cases}\tag{2}$$

where $f^u_x(t)=f_x(t,y^u(t),y^u(t-\delta),z^u(t),\bar z^u(t),u(t))$, $x=y,y_\delta,z,\bar z,v$. Next, setting

$$y^\rho(t)=\rho^{-1}[y^{v^\rho}(t)-y^u(t)]-y_1(t),$$
$$z^\rho(t)=\rho^{-1}[z^{v^\rho}(t)-z^u(t)]-z_1(t),$$
$$\bar z^\rho(t)=\rho^{-1}[\bar z^{v^\rho}(t)-\bar z^u(t)]-\bar z_1(t).$$

Then, we can get the following two lemmas by using Lemma 3.1, the Lebesgue dominated convergence theorem, and Taylor expansion. The technique is classical. Thus, we omit the details and only state the main result for simplicity.