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A revisit to stochastic near-optimal controls: The critical case

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ABSTRACT

This paper revisits the stochastic near-optimal control problem considered in Zhou (1998), where the stochastic system is given by a controlled stochastic differential equation with the control variable taking values in a general control space and entering both the drift and diffusion coefficients. A necessary condition of near-optimality is derived using Ekeland's variational principle, spike variation techniques, and some delicate estimates for the state and the adjoint processes. We improve the error bound of order from "almost" $\varepsilon^{\frac{1}{3}}$ in Zhou (1998) to "exactly" $\varepsilon^{\frac{1}{3}}$.

1. Introduction

In recent years, near optimization has become an important research topic in optimal control theory. Compared with its exact-optimality counterpart, near optimality has many appealing properties, which are useful in both theory and applications. For example, near-optimal controls always exist whereas optimal controls may not exist in many situations; there are many candidates for near-optimal controls, which can be selected easily and appropriately for analysis and implementation. In most practical situations, a near-optimal control suffices to guide decision making, whereas it is usually unrealistic and unnecessary to explore optimal controls that are very sensitive to external perturbations. Interested readers may refer to Ref. [1] for an in-depth discussion aof the merits of near optimality.

Indeed, there have been many studies on near-optimal controls in both deterministic and stochastic cases. Refs. [2,3] investigated near-optimal controls for deterministic dynamical systems. The history of near optimality under stochastic systems can be dated back to Ref. [4], where necessary conditions were derived for some near-optimal controls. Ref. [5] provided a sufficient condition for near-optimal stochastic controls and applied it to general manufacturing systems. Ref. [1] derived necessary and sufficient conditions for all near-optimal controls under forward systems of the diffusion type. Current research focuses on near-optimal controls under various systems. The readers are directed to Ref. [6] for regime-switching systems; Refs. [7,8], and [9] for forward–backward systems; Refs. [10] and [11] for jump-diffusion systems; [12] for recursive systems; and references therein.

Although near optimality has been extensively studied under different stochastic systems, the error bounds in almost all the previous literature, where the control variable can take values in a general (non-convex) space and enter both the drift and diffusion coefficients, are of order "almost" $\varepsilon^{\frac{1}{3}}$ and "exactly" $\varepsilon^{\frac{1}{2}}$ under the necessary and sufficient conditions, respectively. Whether or not the error bounds can be improved is still an open problem. In this paper, we revisit the stochastic near-optimal control problem in Ref. [1]. That is, a near-optimal control problem under forward systems, where the control variable enters both the drift and diffusion coefficients and the control space, is not required to be convex. We consider the critical case of order "exactly" $\varepsilon^{\frac{1}{3}}$ in the necessary condition for near optimality. The aim of this paper is to improve the error bound of order from "almost" $\varepsilon^{\frac{1}{3}}$ to "exactly" $\varepsilon^{\frac{1}{3}}$. The proof of the necessary condition is based on Ekeland's variational principle and the spike variation technique. Borrowing the metric proposed in Ref. [13], we derive some delicate estimates for the state and the adjoint processes. Under certain linear growth and Lipschtiz conditions, we show that any ε-optimal control nearly maximizes the so-called $\mathcal{H}$-function with an error order of "exactly" $\varepsilon^{\frac{1}{3}}$. To our knowledge, this is the best error bound among the existing studies.

The rest of this paper is structured as follows. In Section 2, we introduce basic notation, formulate the near-optimal control

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problem, and present some useful preliminary results. Section 3 is devoted to our main result, that is, a necessary condition for near optimality in the critical case. The final section gives some concluding remarks.

2. Problem formulation and preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, on which is defined a d-dimensional standard Brownian motion $W(\cdot) := \{W(t) | 0 \le t \le T\}$, where $T > 0$ is a finite time horizon. Let $\mathbb{F} := \{\mathcal{F}_t | 0 \le t \le T\}$ be a natural filtration generated by $W(\cdot)$ augmented by all the P-null sets in $\mathcal{F}$. We denote the predictable σ-field on $[0, T] \times \Omega$ by $\mathscr{P}$ and the Borel σ-algebra of any topological space Λ by $\mathscr{B}(\Lambda)$. Let E be a Euclidean space, in which the inner product and the norm is denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively. For a function $\phi : \mathbb{R}^n \to \mathbb{R}$, its gradient is denoted by ϕ_x and its Hessian by ϕ_{xx} (a symmetric matrix). If $\phi : \mathbb{R}^n \to \mathbb{R}^k$, where $k \ge 2$, then $\phi_x = [\frac{\partial \phi_i}{\partial x_j}]_{i=1,2,\ldots,k; j=1,2,\ldots,n}$ is the corresponding $(k \times n)$-Jacobian matrix. In what follows, the transpose of any vector or matrix A is denoted by A^*, and a generic constant by C, which may be different from line to line.

Next, we introduce some spaces of random variables and stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$. For any $\alpha, \beta \in [1, \infty)$, we define

- $L^\beta_{\mathbb{F}}(0, T; E)$: the space of all E-valued, $\mathbb{F}$-adapted processes $\{f(t, \omega) | (t, \omega) \in [0, T] \times \Omega\}$ such that $\|f\|_{L^\beta_{\mathbb{F}}(0,T;E)} := \left\{\mathbb{E}\left[\int_0^T |f(t)|^\beta dt\right]\right\}^{\frac{1}{\beta}} < \infty$;
- $S^\beta_{\mathbb{F}}(0, T; E)$: the space of all E-valued, $\mathbb{F}$-adapted, càdlàg processes $\{f(t, \omega) | (t, \omega) \in [0, T] \times \Omega\}$ such that $\|f\|_{S^\beta_{\mathbb{F}}(0,T;E)} := \left\{\mathbb{E}\left[\sup_{t\in[0,T]} |f(t)|^\beta\right]\right\}^{\frac{1}{\beta}} < \infty$;
- $L^\beta_{\mathcal{F}_T}(\Omega; E)$: the space of all E-valued, $\mathcal{F}_T$-measurable random variables ξ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\|\xi\|_{L^\beta_{\mathcal{F}_T}(\Omega;E)} := \left\{\mathbb{E}\left[|\xi|^\beta\right]\right\}^{\frac{1}{\beta}} < \infty$; and
- $L^\beta_{\mathbb{F}}(0, T; L^\alpha(0, T; E))$: the space of all $L^\alpha(0, T; E)$-valued, $\mathbb{F}$-adapted processes $\{f(t, \omega) | (t, \omega) \in [0, T] \times \Omega\}$ such that $\|f\|_{L^\beta_{\mathbb{F}}(0,T;L^\alpha(0,T;E))} := \left\{\mathbb{E}\left[\left(\int_0^T |f(t)|^\alpha dt\right)^{\frac{\beta}{\alpha}}\right]\right\}^{\frac{1}{\beta}} < \infty.$

Let U denote the control space, which will be specified subsequently. A control process $u(\cdot)$ is said to be admissible if it is a U-valued and $\mathbb{F}$-adapted process. The set of all admissible control processes is denoted by $\mathcal{A}$. For any admissible control $u(\cdot) \in \mathcal{A}$, we consider a stochastic control problem with the controlled stochastic differential equation and the cost functional given by

$$\begin{cases} dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dW(t), \\ X(t) = x_0 \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

and

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^T f(t, X(t), u(t))dt + g(X(T))\right], \tag{2.2}$$

respectively. Here, $b : [0, T] \times \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : [0, T] \times \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}, f : [0, T] \times \Omega \times \mathbb{R}^n \times U \to \mathbb{R}, g : \Omega \times \mathbb{R}^n \to \mathbb{R}$ are given mappings such that b, σ, f are $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^n) \otimes \mathscr{B}(U)$-measurable, and g is $\mathcal{F}_T \otimes \mathscr{B}(\mathbb{R}^n)$-measurable.

Assumption 2.1. The following standing assumptions will be in force throughout this paper:

(i) The control space U is a nonempty compact subset of $\mathbb{R}^k$.

(ii) For each $(x, u) \in \mathbb{R}^n \times U$, $b(\cdot, x, u)$, $\sigma(\cdot, x, u)$ and $f(\cdot, x, u)$ are $\mathbb{F}$-adapted, and $g(x)$ is $\mathcal{F}_T$-measurable.

(iii) For almost all $(t, \omega, u) \in [0, T] \times \Omega \times U$, the mapping

$$x \to (b(t, \omega, x, u), \sigma(t, \omega, x, u), f(t, \omega, x, u), g(\omega, x))$$

is twice differentiable, and the partial derivatives of b, σ, f and g with respect to x up to order 2 is continuous in (x, u). Moreover, there exists a constant $C > 0$ such that for $\phi = b, \sigma, f$ and all $x, x' \in \mathbb{R}^n$, $u \in U$,

$$\begin{cases} |\phi(t, x, u)| \le C(1 + |x| + |u|), \quad |g(x)| \le C(1 + |x|), \\ |\phi(t, x, u) - \phi(t, x', u)| + |\phi_x(t, x, u) \\ \quad - \phi_x(t, x', u)| + |\phi_{xx}(t, x, u) - \phi_{xx}(t, x', u)| \le C|x - x'|, \\ |g(x) - g(x')| + |g_x(x) - g_x(x')| \\ \quad + |g_{xx}(x) - g_{xx}(x')| \le C|x - x'|. \end{cases}$$

Remark 2.1. Our assumptions on the control system are different from those in Ref. [1]. From (i), we can see that the control variable is indeed bounded in $\mathbb{R}^k$. Therefore, the linear growth condition in (iii) can be simplified as

$$|\phi(t, x, u)| \le C(1 + |x|),$$

where $\phi = b, \sigma, f$, which is essentially the same as the linear growth condition in Ref. [1].

Under Assumption 2.1, we can see that for any given admissible control $u(\cdot)$, the stochastic differential equation (2.1) admits a unique solution $X(\cdot) \in S^\beta_{\mathbb{F}}(0, T; \mathbb{R}^n)$, for any $\beta \ge 1$. We call $(X(\cdot), u(\cdot))$ the admissible pair, for any $u(\cdot) \in \mathcal{A}$. In particular, we write $X^u(\cdot)$ for the state process associated with the admissible control $u(\cdot)$ whenever we want to emphasize the dependence of $X(\cdot)$ on $u(\cdot)$. The objective of the control problem is to minimize the cost functional $J(u(\cdot))$, for a given $x_0 \in \mathbb{R}^n$, over all $u(\cdot) \in \mathcal{A}$. Under Assumption 2.1, it is easy to check that $|J(u(\cdot))| < \infty$. Mathematically, the optimal control problem under consideration in this paper is

Problem 2.1.

$$\begin{cases} V(x_0) = \inf_{u(\cdot)\in\mathcal{A}} J(u(\cdot)), \\ \text{subject to } (X(\cdot), u(\cdot)) \text{ satisfies (2.1)}. \end{cases}$$

Here, $V(x_0)$ refers to the value function of Problem 2.1. A control process $\bar{u}(\cdot) \in \mathcal{A}$ is called optimal if it achieves the infimum of $J(u(\cdot))$ over $\mathcal{A}$. The corresponding state process $\bar{X}(\cdot)$ is called the optimal state process. Correspondingly, $(\bar{X}(\cdot), \bar{u}(\cdot))$ is called an optimal pair of Problem 2.1.

As the objective of this paper is to study near-optimal controls rather than exact-optimal ones, we give the precise definitions of near optimality as given in Ref. [1]:

Definition 2.1. For a given $\varepsilon \ge 0$, an admissible pair $(X^\varepsilon(\cdot), u^\varepsilon(\cdot))$ is called ε-optimal, if

$$|J(u^\varepsilon(\cdot)) - V(x_0)| \le \varepsilon.$$

Definition 2.2. Both a family of admissible control pairs $(X^\varepsilon(\cdot), u^\varepsilon(\cdot))$ parameterized by $\varepsilon \ge 0$ and any element $(X^\varepsilon(\cdot), u^\varepsilon(\cdot))$, or simply $u^\varepsilon(\cdot)$, in the family are called near-optimal if

$$|J(u^\varepsilon(\cdot)) - V(x_0)| \le r(\varepsilon)$$

holds for sufficient small ε, where r is a function of ε such that $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. The estimate $r(\varepsilon)$ is called an error bound. If $r(\varepsilon) = C\varepsilon^\delta$ for some $\delta > 0$ independent of the constant C, then $u^\varepsilon(\cdot)$ is called near optimal with the order ε^δ.

Before we conclude this section, let us recall Ekeland's variational principle: