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An algebraic expression of finite horizon optimal control algorithm for stochastic logical dynamical systems

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ABSTRACT

This paper investigates the finite horizon optimal control problem for the stochastic logical dynamical systems with finite states. After giving the equivalent descriptions of stochastic logical dynamical system in term of Markov process, the finite horizon optimization problem is presented in an algebraic form. Based on semi-tensor product of matrix and the increasing dimensional technique, a succinct algebraic expression of dynamic programming algorithm is derived to solve the optimal control problem. Examples, including an application on stochastic Kleene's logical optimization problem, are presented to show the effectiveness of our main result.

1. Introduction

In dynamical systems theory, logical dynamical system has been widely investigated for the last two decades. The essential feature of the logical dynamical system is that the state variable is defined in a logic domain [1], and the logic domain usually consists of a finite or a countably infinite logic states. For such kind of systems, analysis and synthesis under a logical dynamical system framework usually leads to simple logical control law [2,3]. Therefore, the logic-based control becomes a hot topic in control community. Some fundamental issues such as stability, stabilization, controllability, observability, and realization of Boolean or multi-valued logical dynamics can be found in [4,5,2,6,7]. Regarding the random characteristic, logical dynamical systems with stochastic properties have been addressed in several literatures [8–10]. Finite or infinite horizon optimal control problems for stochastic Boolean systems (probabilistic Boolean networks) have been investigated in [11,12]. Furthermore, the application of stochastic logical control has enriched in various fields, including genetic regulatory networks [13], man–machine dynamic game [14] and internal combustion engines control [15].

In this paper, the finite horizon optimal control problem for the stochastic logical dynamical systems with finite state is considered. The purpose of this work is to give a sharp and easy-to-computer-implement algebraic form of finite horizon optimal control algorithm for the stochastic logical dynamical systems. The main contributions of this note are as follows. (i) The dynamics of the stochastic logical system is converted to an algebraic form via the semi-tensor product [16] and Markov process approach. (ii) The algebraic form of value iteration algorithm is presented to solve the optimization problem, by deriving a matrix expression of the dynamic programming based on the increasing dimensional technique.

The rest of this paper is organized as follows. Section 2 is the problem formulation, which introduces some notations and concepts of finite horizon optimal control for stochastic logical dynamical systems. In Section 3, after giving a brief review of semi-tensor product, the matrix expression of the optimization problem is represented for the stochastic logical dynamical system. The main results of this paper are presented in Section 4 and an application on stochastic Kleene's logical optimization problem is given in Section 5. Section 6 is a brief conclusion.

2. Problem formulation

Assume the logic state space consists of finite states, denoted by $S = \{x^1, x^2, \ldots, x^s\}$, and the control space U consists of finite controls, denoted by $U = \{u^1, u^2, \ldots, u^r\}$, respectively.

The most usage way to describe stochastic dynamics with control is modeled by a stationary discrete-time evolution equation

$$x_{k+1} = f(x_k, u_k, w_k), \tag{1}$$

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where $w_k (k = 0, 1, 2, \ldots)$, is external random disturbances, k is the time index. Let $w_k (k = 0, 1, 2, \ldots)$ be independent identical random variable on the disturbance space Ω, and suppose w_k has a conditional distribution $F_W(\cdot|x_k, u_k)$ given current state x_k and current control u_k. The probability of w_k may depend explicitly on x_k and u_k but not on values of prior disturbances $w_{k-1}, \ldots, w_0$.

Consider the class of policies (also called control laws) that consists of a sequence of functions

$$\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}, \tag{2}$$

where $\mu_k : S \to U, k = 0, 1, \ldots, N-1$, maps states x_k into controls $u_k = \mu_k(x_k)$ and in such that $\mu_k(x_k) \in U$ for all $x_k \in S$. Such policies will be called admissible. If an admissible policy $\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$ is given, the stochastic dynamical system (1) become a closed-loop dynamical system from time 0 to time $N-1$ as follow

$$x_{k+1} = f(x_k, \mu_k(x_k), w_k), \tag{3}$$

where the control input $\mu_k(x_k)$ at time k with knowledge of the current state x_k. Given an initial state x_0, and an admissible policy $\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$, define the total expected cost

$$J_\pi(x_0) = \underset{\substack{w_k \\ k=0,1,\ldots,N-1}}{E} \left\{ \mathcal{K}(x_N) + \sum_{k=0}^{N-1} g(x_k, u_k) \right\}, \tag{4}$$

where $g : S \times U \to R$ is the per-step cost function and $\mathcal{K} : S \to R$ is the terminal cost function. We denote by Π the set of all admissible policies π, that is, the set of all sequences of functions $\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$. The optimal cost function J^* is defined by

$$J^*(x_0) = \inf_{\pi \in \Pi} J_\pi(x_0),$$

subject to the stochastic logical dynamical system (1) and initial state $x_0 \in S$.

The finite horizon optimal control problem is to determine an optimal policy $\pi^* \in \Pi$ such that the cost function reaches the optimal J^* under the control of the optimal policy

$$J_{\pi^*}(x_0) = J^*(x_0), \quad \text{for all } x_0 \in S.$$

It should be noted that this problem belong to the category of discrete-time stochastic dynamic programming and might be solved with the backward recursive algorithm based on the principle of optimality. However, a routine work with this kind of algorithm leads to complexity in implantation of the algorithm and increasing of computing loads. In the following, a new algebraic expression of the dynamic programming will be developed that enables one to solve the optimization problem in an algebraic approach and easy-to-programming.

3. Markov process description of stochastic logical dynamics

For convenience of description, the following notations will be used:

(i) The set of $m \times n$ real matrices is denoted by $\mathcal{M}_{m\times n}$.

(ii) Let $N \in \mathcal{M}_{m\times 1}$ and $M \in \mathcal{M}_{m\times n}$. Then $[N]_i$ denotes the ith component of vector N, and $Col_i(M)(Row_i(M))$ denotes the ith column (row) of M, respectively.

(iii) δ_s^i denote the i-column of the identity matrix I_s. And set

$$\Delta_s := \{\delta_s^i | i = 1, 2, \ldots, s\}.$$

(iv) A matrix $L \in \mathcal{M}_{m\times n}$ is called a logical matrix if $Col(L) \subset \Delta_m$, where $Col(L)$ denote the set of columns of L. Then any logical matrix L has the form $L = [\delta_m^{i_1}, \delta_m^{i_2}, \ldots, \delta_m^{i_n}]$, and briefly is defined as

$$L = \delta_m[i_1, i_2, \ldots, i_n].$$

The set of $m \times n$ logical matrices is denoted by $\mathcal{L}_{m\times n}$.

First, we identify the state space S with Δ_s as follows:

$$x^i \sim \delta_s^i, \quad i = 1, 2, \ldots, s. \tag{5}$$

Then, each state $x \in S$ has its corresponding vector form (still use x) $x \in \Delta_s$. Similarly, we identify the control space U with Δ_r as follows: $u^j \sim \delta_r^j$, $j = 1, 2, \ldots, r$. Under the above identification, the per-step cost function $g : S \times U \to R$ can be expressed in the form

$$g(x, u) = x^T G u, \quad \forall x \in \Delta_s, \ u \in \Delta_r,$$

with $G = (G_{i,j})_{s\times r} = \left(g(\delta_s^i, \delta_r^j)\right)_{s\times r}$, and the terminal cost function $\mathcal{K}$ on S can be represented by a s-dimensional vector $\mathcal{K} = \left(\mathcal{K}(\delta_s^1), \ldots, \mathcal{K}(\delta_s^s)\right)^T$.

Semi-tensor product of matrices is a generalization of conventional matrix product [17]. It has been successfully used for both continuous time dynamic systems and discrete time logical dynamic system [16], and many excellent results have been obtained on calculating fixed points and cycles of Boolean network dynamical system [3], the controllability and observability of Boolean control networks [2], and other related issues [18–20].

Definition 3.1 ([16]). Let $M \in \mathcal{M}_{m\times n}$, $N \in \mathcal{M}_{p\times q}$. The semi-tensor product of M and N, denoted as $M \ltimes N$, is defined by

$$M \ltimes N := (M \otimes I_{s/n})(N \otimes I_{s/p}), \tag{6}$$

where $s = lcm\{n, p\}$ is the least common multiple of n and p; $\otimes$ is the Kronecker product.

Remark 3.1. (i) All the properties of conventional matrix product remain true for this generalization. For example, for any $\alpha, \beta \in R$,

(1) Distributive rule: $\begin{cases} A \ltimes (\alpha B + \beta C) = \alpha A \ltimes B + \beta A \ltimes C, \\ (\alpha B + \beta C) \ltimes A = \alpha B \ltimes A + \beta C \ltimes A. \end{cases}$

(2) Associative rule: $A \ltimes (B \ltimes C) = (A \ltimes B) \ltimes C$.

(ii) Some special cases are given here to illustrate the definition of semi-tensor product.

(1) Let $x = [x_1, x_2, \ldots, x_m] \in R^m$, $y = [y_1, y_2, \ldots, y_n] \in R^n$, Then, the semi-tensor product $x \ltimes y$ is

$$[x_1y_1, x_2y_1, \ldots, x_my_1, \ldots, x_1y_n, \ldots, x_my_n] \in R^{mn}.$$

(2) Let X be a row vector of dimension np, and Y be a column vector of dimension p. Then we split into p equal-size blocks as $X^1, \ldots, X^p$, which are $1 \times n$ rows. Then, the semi-tensor product $x \ltimes y$ is $x \ltimes y = \sum_{i=1}^p X^i y_i \in R^n$.

Now we represent the Markov process description of the stochastic logical dynamical system (1). Denote by $p_{ij}(u)$ the transition probabilities from state δ_i to the next state δ_j under control u,

$$p_{ij}(u) = P(x_{k+1} = \delta_s^j | x_k = \delta_s^i, u), \tag{7}$$

for all $\delta_s^i, \delta_s^j \in \Delta_s$, $u \in U$. It is noticed that the transition probabilities $p_{ij}(u)$ satisfy $\sum_{j=1}^s p_{ij}(u) = 1$, $\forall\, i = 1, \ldots, s$, $u \in U$.

Given a discrete-time evolution system in the form (1) together with the probability distribution $F_W\{w_k|x_k, u_k\}$ of w_k, we can provide an equivalent Markov process description. The corresponding transition probability is calculated by

$$p_{ij}(u) = P_W\left(\Omega_{ij}(u) | \delta_s^i, u\right), \tag{8}$$

where $\Omega_{ij}(u)$ is the subset of disturbance space Ω,

$$\Omega_{ij}(u) = \{w : f(\delta_s^i, u, w) = \delta_s^j\}, \quad \forall \delta_s^i, \delta_s^j \in \Delta_s, \ u \in \Delta_r. \tag{9}$$

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