



Numerical null controllability of the 1D linear Schrödinger equation



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ABSTRACT

This paper deals with the numerical approximation to boundary controls that drive the solution to the 1D linear Schrödinger equation to a prescribed state at a final time. Using ideas from Fursikov and Imanuvilov, we consider the control that minimizes over the class of admissible controls a functional that involves weighted integrals, with weights that blow up at T . We will see that this extremal problem is equivalent to a differential problem that is fourth order in space and second order in time. Adapting some numerical techniques applied by the first author and Münch to the heat equation, we approximate the variational formulation by introducing appropriate space–time finite elements that are C^1 in space and C^0 in time. We present two approaches; the second one relies on a change of variable which leads to a lower condition number for the stiffness matrix. The results of some experiments show the efficiency of these methods.

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1. Introduction, the null controllability problem

We are mainly concerned with the boundary exact controllability for the 1D linear Schrödinger equation. The state equation is the following:

$$\begin{cases} i y_t - y_{xx} + V(x, t)y = 0, & (x, t) \in (0, 1) \times (0, T), \\ y(0, t) = u(t), \quad y(1, t) = 0, & t \in (0, T), \\ y(x, 0) = y_0(x), & x \in (0, 1). \end{cases} \quad (1)$$

Here, $T > 0$ and we assume that $y_0 \in H_0^1((0, 1); \mathbb{C})$ and $V, V_x \in L^\infty((0, 1) \times (0, T); \mathbb{R})$. In (1), $u \in L^2((0, T); \mathbb{C})$ is the control and $y = y(x, t)$ is the associated state.

In the sequel, we will use the notation

$$Ly := i y_t - y_{xx} + V y.$$

It is well known that, for any $u \in L^2((0, T); \mathbb{C})$, problem (1) has exactly one solution y in the transposition sense, with

$$y \in C^0([0, T]; H^{-1}((0, 1); \mathbb{C})) \cap H^{-1}(0, T; L^2((0, 1); \mathbb{C})), \quad (2)$$

see for instance [1,2].

Our aim in this paper is to find numerical approximations to controls u such that the associated solutions to (1) satisfy $y(\cdot, T) = 0$. This is called a null controllability problem. In fact, due to the

time reversibility of the linear Schrödinger equation, the null controllability and the exact controllability properties are equivalent, which means that we can reach *any* final state in $H^{-1}((0, 1); \mathbb{C})$ by the action of a boundary control. From now on, we will investigate the null controllability problem.

It is known that, for any $T > 0$, (1) has the null controllability property. In other words, for any $y_0 \in H_0^1((0, 1); \mathbb{C})$, there exist controls $u \in L^2((0, T); \mathbb{C})$ such that the associated states satisfy $y(\cdot, T) = 0$. This was proved in [2] for $V \equiv 0$ by applying the so-called *Hilbert uniqueness method* together with multipliers techniques. In particular, it was established that the control of minimal norm in $L^2((0, T); \mathbb{C})$ is given by $u = \phi_x(0, \cdot)$, where ϕ solves a backward Schrödinger problem

$$\begin{cases} i \phi_t - \phi_{xx} = 0, & (x, t) \in (0, 1) \times (0, T), \\ \phi(x, t) = 0, & (x, t) \in \{0, 1\} \times (0, T), \\ \phi(x, T) = \phi^T(x), & x \in (0, 1), \end{cases}$$

with ϕ^T in an appropriate space.

The null controllability with a time-independent potential V has also been established by other methods. Thus, in Lebeau [3], Hilbert uniqueness was used in combination with microlocal analysis and extended to higher dimensional Schrödinger systems. Later, Tataru [4,5] and Triggiani [6] used appropriate Carleman inequalities to deduce approximate and exact controllability and stabilizability results. Other proofs of controllability have been furnished by Horn and Littman [7,8] and Phung [9].

In the present work, we will use some ideas inspired by the work of Fursikov and Imanuvilov in [10] for similar parabolic systems. More precisely, let us consider the following extremal

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problem:

$$\begin{cases} \text{Minimize } J(y, u) = \frac{1}{2} \iint_Q \rho^2 |y|^2 dx dt \\ \quad + \frac{1}{2} \int_0^T \rho_1(0, t) |u|^2 dt \\ \text{Subject to } (y, v) \in \mathcal{C}(y_0, T). \end{cases} \quad (3)$$

Here and in the sequel, $Q = (0, 1) \times (0, T)$ and $\mathcal{C}(y_0, T)$ is the linear manifold

$$\begin{aligned} \mathcal{C}(y_0, T) &= \{(y, u) \in X : y \text{ solves (1) and satisfies } y(\cdot, T) = 0\} \\ \text{where} \\ X &= L^2(Q; \mathbb{C}) \times L^2((0, T); \mathbb{C}). \end{aligned} \quad (4)$$

We assume that

$$\begin{cases} \rho = \rho(x, t), \\ \rho_1 = \rho_1(x, t) \text{ are continuous, real-valued and } \geq \rho_* > 0, \\ \rho, \rho_1 \in L^\infty((0, 1) \times (0, T - \delta); \mathbb{R}) \quad \forall \delta > 0, \end{cases} \quad (5)$$

so that, in principle, they can blow up as $t \rightarrow T^-$.

The fact that we search for null controls and associated states solving (3) can be justified as follows: first, they can serve to select the “good” control-state pair, according to a previously established criterion; second, they avoid unpleasant oscillations of the control as $t \rightarrow T$ (it is well known that this phenomenon can appear if, for instance, we simply try to find minimal L^2 norm null controls; see [11]).

The main goal in this paper is to solve the extremal problem (3) numerically. To this purpose, we will see that the manifold $\mathcal{C}(y_0, T)$ in (3) is non-empty and (3) possesses exactly one solution.

In the sequel, we will denote by C a positive generic constant and $\langle \cdot, \cdot \rangle$ will stand for the usual duality pairing for H_0^1 and H^{-1} .

The paper is organized as follows. In Section 2, we present two equivalent variational equalities whose solutions p and w furnish the unique solution to (3); see (13) and (18). We will see that the pair (y, u) obtained by the Fursikov–Imanuvilov method belongs to X , which is an interesting additional property, since the natural regularity for y is (2). In Section 3, these variational equalities are analyzed numerically. We introduce some families of approximate problems and we prove appropriate convergence results. Section 4 deals with the results of some numerical experiments. It is seen that the proposed strategies are efficient and furnish satisfactory approximations to the control-state pair (y, u) . Finally, some additional comments are given in Section 5.

2. Variational approaches to the controllability problem

2.1. Preliminaries. A first variational equality

Let us introduce the weights

$$\rho(x, t) \equiv \exp\left(\frac{\alpha(x)}{T-t}\right), \quad \rho_0(x, t) \equiv \rho(x, t)(T-t)^{3/2}, \quad (6)$$

$$\rho_1(x, t) \equiv \rho(x, t)(T-t)^{1/2}, \quad \rho_2(x, t) \equiv \rho(x, t)(T-t)^{-3/2},$$

where

$$\begin{aligned} \alpha(x) &= K_1(e^{K_2} - e^{\beta_0(x)}), \quad \beta_0(x) \equiv \beta_{00}(1-x), \quad K_1 > 0, \\ K_2 &> \beta_{00} > 0. \end{aligned} \quad (7)$$

Obviously, ρ and ρ_1 satisfy (5). Let us consider the extremal problem (3). The roles of ρ and ρ_0 are clarified by the following arguments and results.

Let us set

$$P_0 = \{q \in C^2(\overline{Q}; \mathbb{C}) : q = 0 \text{ on } \{0, 1\} \times [0, T]\}.$$

In this linear space, the sesquilinear form

$$(p, q)_P = \iint_Q \rho^{-2} Lp \overline{Lq} dx dt + \int_0^T \rho_1^{-2}(0, t) p_x(0, t) \overline{q_x(0, t)} dt,$$

is an inner product. This is a consequence of the unique continuation property for the Schrödinger equation; see [1, 12].

Let P be the completion of the space P_0 for the previous inner product. Then, P is a Hilbert space and the following result holds.

Lemma 1. *There exist positive (sufficiently large) constants K_1, K_2 and C_0 such that one has*

$$\begin{aligned} &\iint_Q \rho_2^{-2} |iq_t - q_{xx}|^2 dx dt + \iint_Q \rho_0^{-2} |q|^2 dx dt \\ &\leq C_0 \left(\iint_Q \rho^{-2} |Lq|^2 dx dt + \int_0^T \rho_1^{-2} |q_x(0, t)|^2 dt \right) \end{aligned} \quad (8)$$

for all $q \in P$.

Proof. We can argue as in the proofs of Proposition 1 and Theorem 2 in [1]. Thus, let us introduce the weights

$$\zeta(x, t) \equiv \exp\left(\frac{\alpha(x)}{t(T-t)}\right), \quad \zeta_0(x, t) \equiv \zeta(x, t)(t(T-t))^{3/2},$$

$$\zeta_1(x, t) \equiv \zeta(x, t)(t(T-t))^{1/2},$$

$$\zeta_2(x, t) \equiv \zeta(x, t)(t(T-t))^{-3/2}.$$

We have the following for sufficiently large K_1, K_2, C and C_0 :

$$\begin{aligned} &\iint_Q \zeta_2^{-2} |iq_t - q_{xx}|^2 dx dt + \iint_Q \zeta_1^{-2} |q_x|^2 dx dt \\ &\quad + \iint_Q \zeta_0^{-2} |q|^2 dx dt \\ &\leq C \left(\iint_Q \zeta^{-2} |Lq|^2 dx dt + \int_0^T \zeta_1^{-2} |q_x(0, t)|^2 dt \right) \\ &\leq C_0 \left(\iint_Q \rho^{-2} |Lq|^2 dx dt + \int_0^T \rho_1^{-2} |q_x(0, t)|^2 dt \right). \end{aligned} \quad (9)$$

On the other hand, the usual estimates for the solutions to the Schrödinger equation show that

$$\begin{aligned} &\iint_{(0,1) \times (0,T/2)} |q|^2 dx dt \\ &\leq C \left(\iint_Q \zeta_0^{-2} |q|^2 dx dt + \iint_{(0,1) \times (0,T/2)} |Lq|^2 dx dt \right) \end{aligned}$$

and, taking into account that $iq_t - q_{xx} = Lq - Vq$, we also have

$$\begin{aligned} &\iint_{(0,1) \times (0,T/2)} |iq_t - q_{xx}|^2 dx dt \\ &\leq C \left(\iint_Q \zeta_0^{-2} |q|^2 dx dt + \iint_{(0,1) \times (0,T/2)} |Lq|^2 dx dt \right). \end{aligned}$$

As a consequence, we get (8) for eventually larger constants K_1, K_2 and C_0 . $\square$

As a consequence of Lemma 1, we obtain the following proposition.

Proposition 1. *There exists a unique solution $p \in P$ to the problem*

$$\begin{cases} \iint_Q \rho^{-2} Lp \overline{Lq} dx dt + \int_0^T \rho_1^{-2} p_x(0, t) \overline{q_x(0, t)} dt \\ \quad = i \langle y_0, \overline{q(\cdot, 0)} \rangle \\ \forall q \in P; \quad p \in P. \end{cases} \quad (10)$$

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