Systems & Control Letters 70 (2014) 77–84

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

$\mathcal{H}_\infty$ filter design for nonlinear polynomial systems[☆]

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ARTICLE INFO

Article history:
Received 3 July 2013
Received in revised form
26 May 2014
Accepted 27 May 2014

Keywords:
Polynomial systems
$\mathcal{H}_\infty$ filtering
Linear matrix inequalities

ABSTRACT

The problem of $\mathcal{H}_\infty$ filter design for continuous-time nonlinear polynomial systems is addressed in this paper. The aim is to design a full order dynamic filter that depends polynomially on the filter states. The strategy relies on the use of a quadratic Lyapunov function and an inequality condition that assures an $\mathcal{H}_\infty$ performance bound for the augmented polynomial system, composed by the original system and the filter to be designed, in a regional (local) context. Then, by using Finsler's lemma, an enlarged parameter space is created, where the Lyapunov matrix appears separated from the system matrices in the conditions. Imposing structural constraints to the decision variables and fixing some values for a scalar parameter, design conditions for the $\mathcal{H}_\infty$ filter can be obtained in terms of linear matrix inequalities. As illustrated by numerical experiments, the proposed conditions can improve the $\mathcal{H}_\infty$ performance provided by standard linear filtering by including the polynomial terms in the filter dynamics.

1. Introduction

The filtering problem for linear systems has received a lot of attention in the last years. In the literature, sufficient conditions for the existence of full order filters for uncertain linear systems assuring prescribed $\mathcal{H}_2$ or $\mathcal{H}_\infty$ bounds based on Linear Matrix Inequalities (LMIs) certified through quadratic stability [1–4], parameter dependent Lyapunov functions [5–7] and, more recently, Lyapunov functions with polynomial dependence of degree greater than one [8,9] can be found. In contrast, there are very few results concerned with filter design for systems subject to nonlinearities. In [10] the nonlinearities are assumed to satisfy global Lipschitz conditions and, then, a linear filter is designed by means of LMIs. In [11] a linear $\mathcal{H}_\infty$ filter is proposed for a class of nonlinear systems described by a differential–algebraic representation and [12] tackles the problem of central suboptimal $\mathcal{H}_\infty$ filter design for nonlinear polynomial systems. By applying sum-of-squares (SOS) approaches, [13] proposes a convergent iterative algorithm to solve the problem of linear $\mathcal{H}_\infty$ filters for polynomial systems and [14] addressed the problem of fault detection filter design for nonlinear polynomial plants. In most cases, despite the fact that the system has a nonlinear dynamic model, the implemented filter is linear.

As another aspect of the problem, it is important to underline that the characterization of an estimate of the basin of attraction of the origin for a nonlinear system is a challenging problem [15,16]. Actually, the global stability of the origin can hardly be certified for nonlinear systems in general [17,18]. In [19,20], sufficient conditions for state feedback or observed based control design for quadratic systems are proposed. Additionally, a method to estimate the region of attraction starting from a polytope in the state space is also provided. See, also, the recent work [21] dealing with the convex computation of a region of attraction for polynomial systems based on the use of the theory of moments.

In this paper the problem of $\mathcal{H}_\infty$ filtering for continuous-time nonlinear polynomial systems, i.e., systems whose dynamics depend polynomially on the states, is considered. The filter we want to design has the same structure as the system, i.e., it is a full order dynamic filter with polynomial terms. Firstly, using a quadratic Lyapunov function and LMI based techniques, a sufficient condition that assures an $\mathcal{H}_\infty$ bound to the dynamics of the error system, i.e., the original polynomial system and the proposed filter, in a regional (local) context is obtained. This condition can be viewed as an adaptation of recent results of [22] for state feedback control of saturated quadratic systems. Then, by using Finsler's lemma and imposing structural constraints to the decision variables, quasi-LMI conditions with a scalar parameter are proposed for the design of the matrices of the polynomial filter assuring an $\mathcal{H}_\infty$ bound to the error dynamic system. As illustrated by the numerical experiments, the proposed condition can provide polynomial filters that achieve less conservative $\mathcal{H}_\infty$ bounds when compared to standard

[☆] This work was partially supported by the Brazilian agencies CNPq and São Paulo Research Foundation (FAPESP), grant 2012/04942-8, and CAPES-COFECUB, grant 590/08.

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http://dx.doi.org/10.1016/j.sysconle.2014.05.014
0167-6911/

linear filters. An earlier conference version of the present paper, dealing with the design of quadratic filters for quadratic systems, appeared in [23]. Therefore, the present paper can be viewed as a generalization of [23] to cope with the case of polynomial filters.

The paper is organized as follows. Section 2 presents the system under consideration and the problem we intend to solve. Section 3 provides some preliminary results. The main results are developed in Section 4. Section 5 is dedicated to numerical experiments that illustrate the advantages of the proposed method. Finally, Section 6 concludes the paper.

Notation. The elements of a matrix $A \in \mathbb{R}^{m\times n}$ are denoted by $A_{(i,j)}, i = 1, \dots, m, j = 1, \dots, n$. $A_{(i)}$ denotes the ith row of matrix A. For two symmetric matrices of same dimensions A and B, $A > B$ ($A \geq B$) means that $A - B$ is positive definite (positive semi-definite). For matrices or vectors ($'$) indicates transpose. Matrix $He(Z) = Z + Z'$ is used to simplify the developments. The block-diagonal matrix obtained from vectors is expressed by $diag(x_1, \dots, x_n)$. Similarly, the block-diagonal matrix obtained from matrices, by $diag(X_1, \dots, X_n)$. $\mathscr{L}_2$ represents the Hilbert space of complex signals with finite energy. Identity matrices are denoted by I and null matrices are denoted by 0 of appropriate dimensions. The symbol $\star$ means a symmetric block in matrices.

2. Problem statement

Consider the polynomial nonlinear system of degree g

$$\begin{aligned} \dot{x} &= Ax + \tilde{A}_2x^2 + \tilde{A}_3x^3 + \cdots + \tilde{A}_gx^g + B_1w \\ z &= C_{11}x + \tilde{C}_{12}x^2 + \tilde{C}_{13}x^3 + \cdots + \tilde{C}_{1g}x^g + D_{11}w \\ y &= C_{21}x + \tilde{C}_{22}x^2 + \tilde{C}_{23}x^3 + \cdots + \tilde{C}_{2g}x^g + D_{21}w \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $z \in \mathbb{R}^p$ is the signal to be estimated, $y \in \mathbb{R}^q$ is the measured output and $w \in \mathbb{R}^r$ is the noise input. The signal w is supposed to be energy bounded, that is $w \in \mathscr{L}_2$. Without loss of generality we assume that the signal w is $\mathscr{L}_2$-normalized, that is, it satisfies:

$$\|w\|_2^2 = \int_0^\infty w(\tau)'w(\tau)d\tau \leq 1. \tag{2}$$

The matrices that describe the system have the following dimensions: $A \in \mathbb{R}^{n\times n}, B_1 \in \mathbb{R}^{n\times r}, C_{11} \in \mathbb{R}^{p\times n}, D_{11} \in \mathbb{R}^{p\times r}, C_{21} \in \mathbb{R}^{q\times n}, D_{21} \in \mathbb{R}^{q\times r}$.

Let us present some ingredients in order to write system (1) in another form. The vectors x^i, $i = 1, \dots, g$, represent the vectors with homogeneous terms of degree i, ordered as

$$x^i = \begin{bmatrix} x_1x(1)^{i-1} \\ x_2x(2)^{i-1} \\ \vdots \\ x(n)^i \end{bmatrix} \tag{3}$$

with $x^0 = 1$ and $x(i) = \begin{bmatrix} x_i & \dots & x_n \end{bmatrix}'$, where $x_j, j = 1, \dots, n$, are the components of vector x. Then we have $x^i \in \mathbb{R}^{\sigma_i}$, $\tilde{A}_i \in \mathbb{R}^{n\times\sigma_i}$, $\tilde{C}_{1i} \in \mathbb{R}^{p\times\sigma_i}$ and $\tilde{C}_{2i} \in \mathbb{R}^{q\times\sigma_i}$, $i = 1, \dots, g$, with σ_i given by

$$\sigma_i = \frac{(i + n - 1)!}{i!(n-1)!}. \tag{4}$$

The equivalent representations of terms of degree i are given by

$$\tilde{A}_ix^i = A_iX_ix^{i-1}, \qquad \tilde{C}_{1i}x^i = C_{1i}X_ix^{i-1}, \qquad \tilde{C}_{2i}x^i = C_{2i}X_ix^{i-1} \tag{5}$$

with

$$X_i = \begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{bmatrix} \in \mathbb{R}^{n\sigma_{i-1}\times\sigma_{i-1}} \tag{6}$$

where $A_i \in \mathbb{R}^{n\times n\sigma_{i-1}}$, $C_{1i} \in \mathbb{R}^{p\times n\sigma_{i-1}}$ and $C_{2i} \in \mathbb{R}^{q\times n\sigma_{i-1}}$. System (1) can then be written as follows

$$\begin{aligned} \dot{x} &= (A + A_2X_2)x + (A_3X_3)x^2 + \cdots + (A_gX_g)x^{g-1} + B_1w \\ z &= (C_{11} + C_{12}X_2)x + (C_{13}X_3)x^2 + \cdots + (C_{1g}X_g)x^{g-1} + D_{11}w \\ y &= (C_{21} + C_{22}X_2)x + (C_{23}X_3)x^2 + \cdots + (C_{2g}X_g)x^{g-1} + D_{21}w. \end{aligned} \tag{7}$$

The choice of matrices A_i, C_{1i} and C_{2i} is not unique because the vector $\varphi_i = X_ix^{i-1}$ presents repeated elements. Matrices $\mathbb{I}_i \in \mathbb{R}^{n\sigma_{i-1}\times\sigma_i}$, satisfying $X_ix^{i-1} = \mathbb{I}_ix^i$ define the relation between $\varphi_i = X_ix^{i-1}$ and x^i. An algorithm to construct the matrices $\mathbb{I}_i$ is presented in [24,18]. Let us define vector ϕ containing x^i, $i = 1, \dots, g-1$, as

$$\phi = \begin{bmatrix} x' & x^{2\prime} & \dots & x^{g-1\prime} \end{bmatrix}' \in \mathbb{R}^{\sigma_t} \tag{8}$$

with

$$\sigma_t = \sum_{i=1}^{g-1} \sigma_i. \tag{9}$$

Then one has

$$\begin{aligned} \dot{x} &= \begin{bmatrix} A + A_2X_2 & A_3X_3 & \dots & A_gX_g \end{bmatrix}\phi + B_1w \\ z &= \begin{bmatrix} C_{11} + C_{12}X_2 & C_{13}X_3 & \dots & C_{1g}X_g \end{bmatrix}\phi + D_{11}w \\ y &= \begin{bmatrix} C_{21} + C_{22}X_2 & C_{23}X_3 & \dots & C_{2g}X_g \end{bmatrix}\phi + D_{21}w. \end{aligned} \tag{10}$$

The aim of this work is to find a full-order stable polynomial filter described as

$$\begin{aligned} \dot{x}_f &= \tilde{A}_fx_f + \tilde{A}_{f2}x_f^2 + \tilde{A}_{f3}x_f^3 + \cdots + \tilde{A}_{fg}x_f^g + B_fy \\ z_f &= C_fx_f + D_fy \end{aligned} \tag{11}$$

with $n_f = n, \tilde{A}_f \in \mathbb{R}^{n_f\times n_f}$, $B_f \in \mathbb{R}^{n_f\times q}$, $C_f \in \mathbb{R}^{p\times n_f}$, $D_f \in \mathbb{R}^{p\times q}$, $x_f \in \mathbb{R}^{n_f}$ the estimated state and $z_f \in \mathbb{R}^p$ the estimated output. x_f^i represents the vector with homogeneous terms of degree i, ordered as in (3). Then $x_f^i \in \mathbb{R}^{\sigma_i}$ and $\tilde{A}_{fi} \in \mathbb{R}^{n_f\times\sigma_i}$.

Note that by using similar definitions (5) and (6) with respect to the filter (11), one can write system (11) as

$$\begin{aligned} \dot{x}_f &= (A_f + A_{f2}X_{f2})x_f + (A_{f3}X_{f3})x_f^2 + \cdots + (A_{fg}X_{f3})x_f^{g-1} + B_fy \\ z_f &= C_fx_f + D_fy. \end{aligned} \tag{12}$$

Defining the augmented state vector $\hat{x}^i = \begin{bmatrix} x^{i\prime} & x_f^{i\prime} \end{bmatrix}'$ and the output error $e = z - z_f$, the augmented system reads

$$\begin{aligned} \dot{\hat{x}} &= (\hat{A} + \hat{A}_2\hat{X}_2)\hat{x} + \hat{A}_3\hat{X}_3\hat{x}^2 + \cdots + \hat{A}_g\hat{X}_g\hat{x}^{g-1} + \hat{B}w \\ e &= (\hat{C} + \hat{C}_2\hat{X}_2)\hat{x} + \hat{C}_3\hat{X}_3\hat{x}^2 + \cdots + \hat{C}_g\hat{X}_g\hat{x}^{g-1} + \hat{D}w \end{aligned} \tag{13}$$

where

$$\hat{A} = \begin{bmatrix} A & 0 \\ B_fC_{21} & A_f \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \qquad \hat{A}_i = \begin{bmatrix} A_i & 0 \\ B_fC_{2i} & A_{fi} \end{bmatrix} \in \mathbb{R}^{2n\times 2n\sigma_{i-1}},$$

$$\hat{B} = \begin{bmatrix} B_1 \\ B_fD_{21} \end{bmatrix} \in \mathbb{R}^{2n\times r},$$

$$\hat{C} = \begin{bmatrix} C_{11} - D_fC_{21} & -C_f \end{bmatrix} \in \mathbb{R}^{p\times 2n},$$

$$\hat{C}_i = \begin{bmatrix} C_{1i} - D_fC_{2i} & 0 \end{bmatrix} \in \mathbb{R}^{p\times 2n\sigma_{i-1}},$$

$$\hat{D} = \begin{bmatrix} D_{11} - D_fD_{21} \end{bmatrix} \in \mathbb{R}^{p\times r}, \qquad \hat{X}_i = \begin{bmatrix} X_i & 0 \\ 0 & X_{fi} \end{bmatrix} \in \mathbb{R}^{2n\sigma_{i-1}\times 2\sigma_{i-1}}.$$

At this stage, it could be interesting to study if system (13) with $w = 0$ can be globally asymptotically stable (i.e., asymptotically stable for any initial condition $\hat{x}(0) \in \mathbb{R}^{2n}$). However, the study of the global asymptotic stability for a nonlinear polynomial system is a difficult task, as underlined in [25]. Therefore, it seems that a