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Approximation of distributed delays

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ABSTRACT

In this paper, we address the approximation problem of distributed delays. These elements are convolution operators with kernel having bounded support and appear in the control of time-delay systems. From the rich literature on this topic, we propose a general methodology to achieve such an approximation. For this, we enclose the approximation problem in the graph topology, and working on the convolution Banach algebra, a constructive approximation is proposed. Analysis in time and frequency domains is provided. This methodology is illustrated on the stabilization control for time-delay systems.

1. Introduction

The interest for the use of distributed delays in the stabilization of time-delay systems appears in the pioneering work of Olbrot [1]. To generalize algebraic methods issued from linear systems in finite dimensional spaces to time-delay systems, Kamen et al. [2] first introduce a general mathematical setting for the control, and in particular for the stabilization, of time-delay systems. This mathematical framework was formalized in [3] with the introduction of the Bézout ring of pseudopolynomials. In all these works, distributed delays appear to be central [4]. A distributed delay is a linear input–output convolution operator of the form

$$y(t) = (f * u)(t) = \int_0^{\vartheta} f(\tau)u(t-\tau)\,\mathrm{d}\tau \qquad (1)$$

where $\vartheta > 0$ is bounded, and the kernel $f(\cdot)$ is a continuous function with support $[0, \vartheta]$. Numerical implementation of distributed delay was early investigated. A first proposition for approximation with finite dimensional systems was proposed in [5]. Reduction and approximation of delay systems, involving lumped delays, were also investigated in [6]. In the work of [7], the authors propose a numerical integral approximation to realize an operator as given in (1). Such an approximation writes as a sum of lumped delayed distributions and unfortunately introduces additional closed-loop poles and also instability phenomena. See *e.g.* [8] and references therein. To overcome this problem, various solutions were proposed. An additional low-pass filter in the integral approximation was proposed in [9,10]. Further implementation improvements were made in [11], with rational approximation and extension of bilinear transformations, and in [12], where a shift-based method for rational approximation using the Von Neumann inequality and Padé approximation was analyzed. These last papers address positively the continuous-time approximation of distributed delays. Proposals for a discrete-time realization of distributed delays are included in [13].

A continuous time approximation needs to reproduce with high fidelity the internal dynamics of this operator, for large classes of input signals, and also to generate an arbitrarily close input–output behavior to the original one. To fulfill these objectives, we introduce the kernel approximation that can be realized by various classes of operators, like polynomials, rational fractions, or exponentials. See, *e.g.* [14,15] and references therein. With the objective to substitute the distributed delay by a more tractable system, we propose two classes that realize approximation, namely lumped systems and a subclass of distributed delays. We enclose the approximation problem into the Wiener algebra of BIBO-stable systems, using the graph topology. This corresponds to the weakest topology for which feedback stabilization is a robust property. Moreover, for stable and strictly proper systems,

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graph topology and norm topology being the same, we work on L_1-norm convergence over this algebra. This general framework was used first in [16] for the approximation of distributed parameter systems by lumped systems. This idea also grows in [17], for the approximation of lumped delayed distributions which appear in optimal control. We show that working in this general setting leads to an approximation of the distributed delay in both time and frequency domains, for large classes of input signals.

The paper is organized as follows. In Section 2, we define and characterize the main properties of distributed delays, using decompositions on the so-called elementary distributed delays. In Section 3, we explicit our approximation problem and solve it. A constructive approximation is described in Section 4. Simulations show the effectiveness of the method on the stabilization control problem.

2. Convolution operators and distributed delays

2.1. Convolution algebra

Input–output causal convolution systems described by (1) are naturally defined over a commutative algebra. A general algebra of distributions including a wide class of convolution systems was introduced in [18]. For our purpose, we consider a subalgebra of the Callier–Desoer algebra, and we denote it by $\mathscr{A}$. We say that $f \in \mathscr{A}$ if

$$f(t) = \begin{cases} f_{\mathrm{a}}(t) + f_{\mathrm{pa}}(t), & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{2}$$

where the complex-valued function $f_{\mathrm{a}}(\cdot) \in L_1(\mathbb{R}_+)$, that is f_{a} is a complex valued function, locally integrable on $\mathbb{R}_+$, and such that $\int_0^\infty |f_{\mathrm{a}}(t)|\, \mathrm{d}t < \infty$. The complex-valued distribution f_{pa} stands for the purely atomic part

$$f_{\mathrm{pa}}(t) = \sum_{n=0}^{\infty} f_n \delta(t - t_n), \tag{3}$$

with $f_n \in \mathbb{C}$, $n = 0, 1, \ldots$, $0 = t_0 < t_1 < t_2 < \cdots$, $\delta(t - t_n)$ denoting the Dirac delta distribution centered in t_n, and $\sum_{n \geq 0} |f_n| < \infty$. As shown in Desoer and Vidyasagar [19], $\mathscr{A}$ is a commutative convolution Banach algebra for the norm

$$\|f\|_{\mathscr{A}} = \|f_{\mathrm{a}}\|_{L_1} + \sum_{n=0}^{\infty} |f_n|, \tag{4}$$

and with a unit element the Dirac delta distribution δ. Let us recall the concept of bounded input–bounded output stability.

Definition 1. A convolution system in the form (1) is said to be BIBO stable if $f \in \mathscr{A}$.

Denoting $\hat{f}$ as the Laplace transform of f, $\hat{\mathscr{A}}$ is the set of Laplace transforms of elements in $\mathscr{A}$. It is also a commutative Banach algebra with unit element under pointwise addition and multiplication, for the norm $\left\|\hat{f}\right\|_{\hat{\mathscr{A}}} = \|f\|_{\mathscr{A}}$.

2.2. Distributed delays

Let $\mathbb{I}_{a,b} = [a, b]$ be the bounded closed interval in $\mathbb{R}_+$, for some reals a and b, $0 \leq a < b$. Notations $\mathbb{I}_{0,\infty}$ or $\mathbb{R}_+$ stand for $[0, \infty[$. We define $\mathscr{K}(\mathbb{I}_{a,b})$ as the set of complex valued functions $g(\cdot)$ in the form

$$g(t) = \begin{cases} g_{\mathbb{I}_{a,b}}(t), & t \in \mathbb{I}_{a,b} \\ 0, & \text{elsewhere} \end{cases} \tag{5}$$

where

$$g_{\mathbb{I}_{a,b}}(t) = \sum_{i \geq 0} \sum_{j \geq 0} c_{ij}\, t^j\, \mathrm{e}^{\lambda_i t}, \tag{6}$$

for some c_{ij} and λ_i in $\mathbb{C}$, and the sums are finite. For any real valued function in $\mathscr{K}(\mathbb{I}_{a,b})$, if some $\lambda_i \in \mathbb{C}$ appears in the sum, then so does its conjugate $\bar{\lambda}_i$, and the associated coefficients c_{ij} are complex conjugates. Hence, any real valued function in $\mathscr{K}(\mathbb{I}_{a,b})$ is a function generated by real linear combinations of $t^j \mathrm{e}^{\sigma_i t}$, $t^j \mathrm{e}^{\sigma_i t} \sin(\beta_k t)$ and $t^j \mathrm{e}^{\sigma_i t} \cos(\beta_k t)$, for some reals σ_i, β_k, the sums being finite. The formal definition of distributed delay is as follows.

Definition 2. A distributed delay is a causal convolution system in the form (1) with kernel f in $\mathscr{K}(\mathbb{I}_{\vartheta_1,\vartheta_2})$, for some bounded real numbers $0 \leq \vartheta_1 < \vartheta_2$.

The set of distributed delays is denoted by $\mathscr{G}$. Any distributed delay in $\mathscr{G}$ admits a Laplace transform, corresponding to the finite Laplace transform of its kernel $f \in \mathscr{K}(\mathbb{I}_{\vartheta_1,\vartheta_2})$,

$$\hat{y}(s) = \hat{f}(s)\hat{u}(s), \quad \hat{f}(s) = \int_{\vartheta_1}^{\vartheta_2} f_{\mathbb{I}_{\vartheta_1,\vartheta_2}}(\tau)\, \mathrm{e}^{-s\tau}\, \mathrm{d}\tau, \tag{7}$$

where $\hat{f} \in \hat{\mathscr{G}}$ is an entire function. The notion of elementary distributed delay will greatly simplify the approximation problem. Let us define the complex valued function $\theta_\lambda(\cdot) \in \mathscr{K}(\mathbb{I}_{0,\vartheta})$, for some $\lambda \in \mathbb{C}$ and $\vartheta > 0$, by

$$\theta_\lambda(t) = \begin{cases} \mathrm{e}^{\lambda t}, & t \in [0, \vartheta] \\ 0, & \text{elsewhere} \end{cases} \tag{8}$$

and its Laplace transform

$$\hat{\theta}_\lambda(s) = \frac{1 - \mathrm{e}^{-(s-\lambda)\vartheta}}{s - \lambda}, \tag{9}$$

which is an entire function even at $s = \lambda$ where $\hat{\theta}_\lambda(\lambda) = \vartheta$. In other words, λ is a removable singularity, and consequently $\hat{\theta}_\lambda(s)$ has no pole. The distributed delay with kernel θ_λ is called an elementary distributed delay. The kth derivative $\hat{\theta}_\lambda^{(k)}(s)$ of $\hat{\theta}_\lambda(s)$ leads to

$$\hat{\theta}_\lambda^{(k)}(s) = \int_0^{\vartheta} (-\tau)^k \mathrm{e}^{-(s-\lambda)\tau}\, \mathrm{d}\tau, \tag{10}$$

which is still in $\hat{\mathscr{G}}$ and corresponds to the Laplace transform of the function $\theta_\lambda^k(t) = (-t)^k \theta_\lambda(t)$. From previous definitions, we can state the following lemma, which also appeared in [3], and which plays a central role for approximation.

Lemma 1. *Any element in $\hat{\mathscr{G}}$ can be decomposed into a finite sum of the Laplace transform of elementary distributed delays and their successive derivatives.*

Proof. Take any element in $\mathscr{G}$. Its kernel $g(\cdot)$ lies in $\mathscr{K}(\mathbb{I}_{\vartheta_1,\vartheta_2})$ and writes as in (6). By time translation corresponding to the lumped delay ϑ_1, it is readily a linear finite combination of elementary distributed delays $\theta_\lambda(t)$ and of the functions $\theta_\lambda^k(t)$, as defined in (8) and (10), with $\vartheta = \vartheta_2 - \vartheta_1$. □

From this result, for any $\hat{g} \in \hat{\mathscr{G}}$, there exist complex polynomials $\hat{g}_{ik} \in \mathbb{C}[\mathrm{e}^{-\vartheta s}]$ with respect to the variable $\mathrm{e}^{-\vartheta s}$ and $\lambda_i \in \mathbb{C}$, in finite number, such that

$$\hat{g}(s) = \sum_{i,k} \hat{g}_{ik}(\mathrm{e}^{-\vartheta s}) \hat{\theta}_{\lambda_i}^{(k)}(s), \tag{11}$$

where successive derivatives are iteratively computed by

$$\hat{\theta}_{\lambda_i}^{(k)}(s) = (-1)^k k! \frac{1 - \mathrm{e}^{-(s-\lambda_i)\vartheta} - \sum_{n=1}^{k} \frac{\vartheta^n}{n!} \mathrm{e}^{-(s-\lambda_i)\vartheta} (s - \lambda_i)^n}{(s - \lambda_i)^{k+1}}, \tag{12}$$