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Stabilizability, representations and factorizations for time-varying linear systems[☆]

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ABSTRACT

We study the system representations, factorizations and stabilizability of a discrete-time time-varying linear system in the framework of nest algebra. We shown that every time-varying linear system admits a normalized left factorization, while it may possibly not have a right factorization. The relationship between the system representation and factorization is studied, and some new stabilizability criteria are established.

1. Introduction

Stabilizability is a basic concept for stability analysis of linear systems. The theory on the connection between coprime factorization and different forms of stabilizability of finite-dimensional systems matured in the 1980s [1–3]. Thereafter, coprime factorization played a major role in stability theory for both finite- and infinite-dimensional systems. Smith showed that every stabilizable MIMO system has a coprime fraction representation over H^∞ [3], for which the system is defined by means of a transfer matrix with entries in the quotient field of a certain integral domain of SISO stable plants. Some related results were established for general non-rational functions in the operator-valued H^∞ case [4–6]. With the development of H^∞ control theory, much insight has been obtained by considering the time-varying analog on an appropriate complex Hilbert space of input–output signals. The approach taken in this case is the input–output point: the system is considered as a linear causal operator (possibly unbounded) defined on the separable Hilbert space, and the set of stable, causal, time-varying linear systems is referred to in the literature as the nest algebra. In the case of a discrete-time time-varying linear system, a strong representation as an alternative but equivalent framework to coprime factorization was developed by Dale and Smith [7]. In such situations, a system is stabilizable if and only if it has strong left and right representations. This result allows application of coprime factorization theory to any stabilizable time-varying system, for which there is a Youla-type parameterization theorem that is conceptually similar to the classical result for time-invariant linear systems. Equivalence between the existence of left and right coprime factorizations has been derived according to the complete finiteness of the nest algebra [8,9].

The Youla parameterization was developed for stabilizable systems that admit doubly coprime factorizations. However, this result is generally not true for infinite-dimensional linear systems (e.g. differential time-delay systems, partial differential equations) [10]. Quadrat applied lattice theory in the fractional representation approach to exhibit the general parameterization of all the stabilizing controllers for a stabilizable plant that does not necessarily admit doubly coprime factorizations [11,12]. This parameterization is in fact in terms of weakly coprime factorizations. Weak coprimeness is in many ways a more natural extension of coprimeness to infinite-dimensional systems than the Bezout case. In the finite-dimensional case, coprime factorization of a system is determined by the linear quadratic (LQ) optimal-state feedback. A related result was extended to infinite-dimensional time-invariant systems over arbitrary Hilbert spaces, in the sense that the factorization is weakly coprime [13]. Obviously, factorization is a more natural definition than (weakly) coprime factorization. In addition, a representation is more natural than a strong representation [14,15]. There are some results for factorizations and representations of time-invariant linear systems. Left and

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right factorizations for every finite-dimensional time-variant system always exist. Mikkola showed that this does not hold for infinite-dimensional time-invariant systems [13]. The existence of normalized right representations for finite-dimensional time-invariant linear systems is obtained as a well-known consequence of the Beurling–Lax theorem. Mikkola showed that an infinite-dimensional time-invariant system has a normalized right factorization that is also a right representation if it has a right factorization [13]. However, the same problem has never been solved in the infinite-dimensional case. To the best of our knowledge, there are no corresponding results in the literature for discrete-time time-varying linear systems in the framework of nest algebra. The existence of left and right factorizations for time-varying linear system requires clarification. It is also of great interest to study the representations and factorizations and the relationship between them.

Here we consider properties of the representations, factorizations, and stabilizability for time-varying linear systems. We use the framework described by Dale and Smith [7], which allows us to give some purely operator-theoretic formulations for the stabilizability. Davidson provided a more general framework based on operator algebra methods, in particular nest algebra [16]. We are not concerned with systems in state–space form [6,13], but concentrate on the input–output properties and issues such as the existence of system representations and factorizations. First, we provide an example of a time-varying linear system that does not admit a right factorization. We use operator theory to prove that every time-varying linear system admits a normalized left factorization, and the normalized left factorization is unique modulo the left multiplication by a unitary operator in the nest algebra. We clarify some relationships between the factorizations and representations. We also develop a time-varying generalization of weakly coprime factorization and show that a right factorization is a right representation if and only if it is weakly coprime. Finally, we revisit the feedback stabilization problem and establish some stabilizability criteria in terms of normalized left factorization.

The remainder of the paper is organized as follows. Some notations, definitions, and mathematical results from operator theory and system theory are gathered in Section 2. In Section 3, we study the properties of system representations and factorizations in detail. In Section 4, we derive some necessary and sufficient conditions for the stabilizability of time-varying linear systems. The paper ends with an example.

2. Basic notations and operator theory

In this section, we introduce notations, definitions, and some results used throughout the paper. More details can be found in the literature [15–18].

Let $\mathcal{H}$ and $\mathcal{K}$ be two separable Hilbert spaces. The symbol "$\oplus$" denotes the direct sum of two Hilbert spaces. $\mathcal{B}(\mathcal{H},\mathcal{K})$ denotes the Banach space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, and $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$. The image and kernel of $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ are denoted by $\operatorname{Im} T = \{y \in \mathcal{K} : y = Tx,\ x \in \mathcal{H}\}$ and $\operatorname{Ker} T = \{x \in \mathcal{H} : Tx = 0\}$, respectively. The restriction of T to the closed subspace $\mathcal{V} \subseteq \mathcal{H}$ is denoted by $T|\mathcal{V}$, i.e., $(T|\mathcal{V})(x) = \begin{cases} Tx, & x \in \mathcal{V} \\ 0, & x \in \mathcal{V}^{\perp} \end{cases}$. The adjoint of the operator T is denoted by T^*.

An operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is an isometry if $\|Tx\|_{\mathcal{K}} = \|x\|_{\mathcal{H}}$ for all $x \in \mathcal{H}$; equivalently, $T^*T = I$. T is a co-isometry if T^* is an isometry. $T \in \mathcal{B}(\mathcal{H})$ is a unitary operator if $T^*T = TT^* = I$ [17, Section II.2.17].

Definition 2.1 (*[17, Section IX.1.3]*)**.** Let $\{A_n\}$ be a bounded sequence in $\mathcal{B}(\mathcal{H},\mathcal{K})$.

1. $\{A_n\}$ converges to A in the weak operator topology, denoted by $A_n \xrightarrow{WOT} A$, if $\lim_{n\to\infty}\langle A_n h, k\rangle_{\mathcal{K}} = \langle Ah, k\rangle_{\mathcal{K}}$ for every $h \in \mathcal{H}$, $k \in \mathcal{K}$.
2. $\{A_n\}$ converges to A in the strong operator topology, denoted by $A_n \xrightarrow{SOT} A$, if $\lim_{n\to\infty} A_n h = Ah$ for every $h \in \mathcal{H}$.
3. $\{A_n\}$ converges to A in the weak-star topology if $\lim_{n\to\infty} tr(A_nB) = tr(AB)$ for every trace class operator B in $\mathcal{B}(\mathcal{K},\mathcal{H})$.

Proposition 2.1 (*[18, Chapter 3, Section 20.1]*)**.** (a) *A bounded sequence in $\mathcal{B}(\mathcal{H},\mathcal{K})$ converges in the weak-star topology if and only if it converges in the weak operator topology.*

(b) *The closed unit ball of $\mathcal{B}(\mathcal{H},\mathcal{K})$ is a compact metric space in the weak-star topology.*

Proposition 2.2 (*[19, Chapter 3, Section 28.2]*)**.** *For a metric space X, the followings are equivalent:*

1. *X is compact.*
2. *X is sequentially compact.*

The input–output signal space considered here is a generalization of ℓ^2-sequences to the sequences for which the components have non-uniform dimensions. Given the index sequence $\mathrm{K} = (k_0, k_1, \ldots)$ that is an ordered series of non-zero natural numbers, the signal space is chosen as

$$\ell^{2,\mathrm{K}} := \left\{ x = (x_0, x_1, \ldots, x_n, x_{n+1}, \ldots) : x_i \in \mathbb{C}^{k_i},\ \sum_{i=0}^{+\infty} \|x_i\|^2_{\mathbb{C}^{k_i}} < +\infty \right\}.$$

$\ell^{2,\mathrm{K}}$ is a complex separable Hilbert space with inner product and norm in the following form:

$$\langle x, y\rangle = \sum_{i=0}^{+\infty} \langle x_i, y_i\rangle_{\mathbb{C}^{k_i}}, \qquad \|x\| = \left(\sum_{i=0}^{+\infty} \|x_i\|^2_{\mathbb{C}^{k_i}}\right)^{\frac{1}{2}}.$$

The extended space [15, Chapter 5] of $\ell^{2,\mathrm{K}}$ is defined by

$$\ell_e^{2,\mathrm{K}} := \left\{ x = (x_0, x_1, \ldots, x_n, x_{n+1}, \ldots) : x_i \in \mathbb{C}^{k_i} \right\}.$$

For each $n \geq 0$, P_n denotes the truncation projection on $\ell^{2,\mathrm{K}}$ and $\ell_e^{2,\mathrm{K}}$ defined by

$$P_n(x_0, x_1, \ldots, x_n, x_{n+1}, \ldots) = (x_0, x_1, \ldots, x_n, 0, \ldots).$$

We denote $P_{-1} = 0$ and $P_{+\infty} = I$. For each $n \geq -1$ ($n \neq +\infty$), the seminorm $\|\cdot\|_n$ on $\ell_e^{2,\mathrm{K}}$ is defined by

$$\|x\|_n = \|P_n x\|, \quad x \in \ell_e^{2,\mathrm{K}}.$$

The family of seminorms $\{\|\cdot\|_n : -1 \leq n < +\infty\}$ on vector space $\ell_e^{2,\mathrm{K}}$ satisfies $\bigcap_{-1\leq n<+\infty}\{x : \|x\|_n = 0\} = \{0\}$. Thus, $\ell_e^{2,\mathrm{K}}$ is a locally convex topology space whose topology is defined by $\{\|\cdot\|_n : n \geq -1\}$, called the resolution topology. These lead to a nice characterization of $\ell^{2,\mathrm{K}}$ in $\ell_e^{2,\mathrm{K}}$:

$$\ell^{2,\mathrm{K}} = \left\{ x \in \ell_e^{2,\mathrm{K}} : \sup_{n\geq -1} \|x\|_n < +\infty \right\}.$$

Definition 2.2 (*[15, Chapter 5]*)**.** Let L be a linear transformation on $\ell_e^{2,\mathrm{K}}$.

1. L is causal if $P_nLP_n = P_nL$ for all $0 \leq n < +\infty$.
2. L is a time-varying linear system if L is a causal linear transformation that is continuous with respect to the resolution topology.
3. L is a stable linear system if L is causal and $L|\ell^{2,\mathrm{K}}$ is a bounded linear operator.