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On multipliers for bounded and monotone nonlinearities ☆

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ABSTRACT

Recent results in equivalence between classes of multipliers for slope-restricted nonlinearities are extended to multipliers for bounded and monotone nonlinearities. This extension requires a slightly modified version of the Zames–Falb theorem and a more general definition of phase-substitution. The results in this paper resolve apparent contradictions in the literature on classes of multipliers for bounded and monotone nonlinearities.

1. Introduction

Different classes of multipliers can be used for analysing the stability of a Lur'e system (see Fig. 1) where the nonlinearity is bounded and monotone. A loop transformation allows us to analyse slope-restricted nonlinearities with the same classes of multipliers [1]. Apparently contradictory results can be found in the literature with respect to which class provides better results. On the one hand, it is stated that a complete search over the class of Zames–Falb multipliers will provide the best result that can be achieved [2,3]. On the other hand, searches over a subclass of Zames–Falb multipliers [4,5] have been improved by adding a Popov multiplier [6–8].

The class of Zames–Falb multipliers is formally given in the celebrated paper [1]. Two main results are given: Theorem 1 in [1] presents the Zames–Falb multipliers for bounded and monotone nonlinearities; Corollary 2 in [1] applies the Zames–Falb multipliers to slope-restricted nonlinearities via a loop transformation. We have formally shown in [9] that the class of Zames–Falb multipliers for slope-restricted nonlinearities, i.e. using Corollary 2 in [1], should provide the best result in comparison with any other class of multipliers available in the literature. The result relies on the fact that only biproper plants need to be considered in the search for a Zames–Falb multiplier, since the original plant becomes biproper after the loop transformation in Fig. 2 [1,10].

However, for bounded and monotone nonlinearities, biproperness of the LTI system G cannot be assumed without loss of generality. But the conditions of Theorem 1 in [1] cannot hold when the plant is strictly proper. An example has been proposed in [11] where the addition of a Popov multiplier to the Zames–Falb multiplier is essential to guarantee the stability of the Lur'e system. This prompts the natural question: is the addition of a Popov multiplier an improvement over the class of Zames–Falb multipliers for bounded and monotone nonlinearities? In fact, we show that this restriction of the conditions of Theorem 1 in [1] leads to more fundamental contradictions.

This paper proposes a slightly modified version of Theorem 1 in [1] in such a way that strictly proper plants can be analysed. Then, generalizations of phase-substitution and phase-containment defined in [9] are given in order to show the relationship between classes of multipliers. As a result, we show that a search over the class of Zames–Falb multipliers is also sufficient for bounded and monotone nonlinearities, i.e. if there is no suitable Zames–Falb multiplier then there is no suitable multiplier within any other class of multipliers. This paper resolves some apparent paradoxes, providing consistency to results in the literature.

The structure of the paper is as follows. Section 2 gives preliminary results; in particular, the equivalence results in [9] are stated and the differences between the cases of slope-restricted and bounded and monotone nonlinearities are highlighted. Section 3 provides the relationships between classes for the case of bounded and monotone nonlinearities. Section 4 analyses the example given in [11], showing that there exists a Zames–Falb multiplier that provides the stability result under our modification

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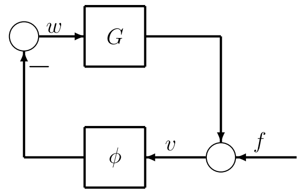

Fig. 1. Lur'e system.

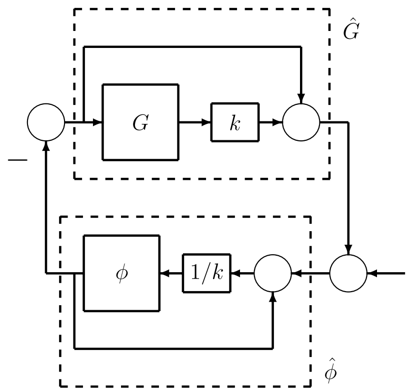

Fig. 2. Loop shifting transforms a slope restricted nonlinearity ϕ into a monotone nonlinearity $\hat{\phi}$. Simultaneously, a new linear system $\hat{G}$ is generated. In [9], we have shown that when generated via loop shifting $\hat{G}$ can be assumed biproper without loss of generality from the necessity of the Kalman conjecture (for further discussion, see Section 2.3 in [9]), but such an assumption cannot be made when there is no loop shifting.

of Theorem 1 in [1]. Finally, the conclusions of this paper are given in Section 5.

2. Notation and preliminary results

Let $\mathcal{L}_2^m[0,\infty)$ be the Hilbert space of all square integrable and Lebesgue measurable functions $f : [0,\infty) \to \mathbb{R}^m$. Similarly, $\mathcal{L}_2^m(-\infty,\infty)$ can be defined for $f : (-\infty,\infty) \to \mathbb{R}^m$. Given $T \in \mathbb{R}$, a truncation of the function f at T is given by $f_T(t) = f(t)\,\forall t \leq T$ and $f_T(t) = 0\,\forall t > T$. The function f belongs to the extended space $\mathcal{L}_{2e}^m[0,\infty)$ if $f_T \in \mathcal{L}_2^m[0,\infty)$ for all $T > 0$. In addition, $\mathcal{L}_1(-\infty,\infty)$ (henceforth $\mathcal{L}_1$) is the space of all absolute integrable functions; given a function $h : \mathbb{R} \to \mathbb{R}$ such that $h \in \mathcal{L}_1$, its $\mathcal{L}_1$-norm is given by

$$\|h\|_1 = \int_{-\infty}^{\infty} |h(t)|\,dt. \tag{1}$$

A nonlinearity $\phi : \mathcal{L}_{2e}[0,\infty) \to \mathcal{L}_{2e}[0,\infty)$ is said to be memoryless if there exists $N : \mathbb{R} \to \mathbb{R}$ such $(\phi v)(t) = N(v(t))$ for all $t \in \mathbb{R}$. Henceforward we assume that $N(0) = 0$. A memoryless nonlinearity ϕ is said to be bounded if there exists a positive constant C such that $|N(x)| < C|x|$ for all $x \in \mathbb{R}$. The nonlinearity ϕ is said to be monotone if for any two real numbers x_1 and x_2 we have

$$0 \leq \frac{N(x_1) - N(x_2)}{x_1 - x_2}. \tag{2}$$

The nonlinearity ϕ is said to be odd if $N(x) = -N(-x)$ for all $x \in \mathbb{R}$.

This paper focuses the stability of the feedback interconnection of a proper stable LTI system G and a bounded and monotone nonlinearity ϕ, represented in Fig. 1 and given by

$$\begin{cases} v = f + Gw, \\ w = -\phi v. \end{cases} \tag{3}$$

Since G is a stable LTI system, the exogenous input in this part of the loop can be taken as the zero signal without loss of generality. It is well-posed if the map $(v, w) \mapsto (0, f)$ has a causal inverse on $\mathcal{L}_{2e}^2[0,\infty)$; this interconnection is $\mathcal{L}_2$-stable if for any $f \in \mathcal{L}_2[0,\infty)$, then $Gw \in \mathcal{L}_2[0,\infty)$ and $\phi v \in \mathcal{L}_2[0,\infty)$, and it is absolutely stable if it is $\mathcal{L}_2$-stable for all ϕ within the class of nonlinearities. In addition, $G(j\omega)$ means the transfer function of the LTI system G. Finally, given an operator M, then M^* means its $\mathcal{L}_2$-adjoint (see [12] for a definition). For LTI systems, $M^*(s) = M^\top(-s)$, where $^\top$ means transpose.

The standard notation $\mathbf{L}_\infty$ ($\mathbf{RL}_\infty$) is used for the space of all (proper real rational) transfer functions bounded on the imaginary axis and infinity; $\mathbf{RH}_\infty$ ($\mathbf{RH}_2$) is used for the space of all (strictly) proper real rational transfer functions such that all their poles have strictly negative real parts; and $\mathbf{RH}_\infty^-$ is used for the space of all proper real rational transfer functions such that all their poles have strictly positive real parts. Moreover, the subset of $\mathbf{RH}_2$ with positive DC gain is referred to as $\mathbf{RH}_2^+$. The H_∞-norm of a SISO transfer function G is defined as

$$\|G\|_\infty = \sup_{\omega\in\mathbb{R}}(|G(j\omega)|). \tag{4}$$

With some acceptable abuse of notation, given a rational strictly proper transfer function $H(s)$ bounded on the imaginary axis, $\|H\|_1$ means the $\mathcal{L}_1$-norm of the impulse response of $H(s)$.

2.1. Zames–Falb theorem and multipliers

The original Theorem 1 in [1] can be stated as follows:

Theorem 2.1 (*[1]*). *Consider the feedback system in Fig. 1 with $G \in \mathbf{RH}_\infty$, and a bounded and monotone nonlinearity ϕ. Assume that the feedback interconnection is well-posed. Then suppose that there exists a convolution operator $M : \mathcal{L}_2(-\infty,\infty) \to \mathcal{L}_2(-\infty,\infty)$ whose impulse response is of the form*

$$m(t) = \delta(t) - \sum_{i=0}^{\infty} z_i\delta(t - t_i) - z_a(t), \tag{5}$$

where δ is the Dirac delta function and

$$\sum_{i=0}^{\infty} |z_i| < \infty, \quad z_a \in \mathcal{L}_1,\ \text{and}\ t_i \in \mathbb{R}\,\forall i \in \mathbb{N}. \tag{6}$$

Assume that:

(i)

$$\|z_a\|_1 + \sum_{i=0}^{\infty} |z_i| < 1, \tag{7}$$

(ii) *either ϕ is odd or $z_a(t) > 0$ for all $t \in \mathbb{R}$ and $z_i > 0$ for all $i \in \mathbb{N}$, and*

(iii) *there exists $\delta > 0$ such that*

$$\operatorname{Re}\{M(j\omega)G(j\omega)\} \geq \delta \quad \forall\omega \in \mathbb{R}. \tag{8}$$

Then the feedback interconnection (3) is $\mathcal{L}_2$-stable. ■

Eqs. (5)–(7) in Theorem 2.1 provide the class of Zames–Falb multipliers. It is a subset of $\mathbf{L}_\infty$, i.e. it is not limited to rational transfer functions. However, for the remainder of this paper we restrict our attention to such rational multipliers, i.e. we set $z_i = 0$ for all $i \in \mathbb{N}$.

Definition 2.2. The class of SISO rational Zames–Falb multipliers $\mathcal{M}$ contains all SISO rational transfer functions $M \in \mathbf{RL}_\infty$ such that $M(s) = 1 - Z(s)$, where $Z(s)$ is a rational strictly proper transfer function and $\|Z\|_1 < 1$.