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Boundary feedback stabilization of the telegraph equation: Decay rates for vanishing damping term

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ABSTRACT

We study a semilinear mildly damped wave equation that contains the telegraph equation as a special case. We consider Neumann velocity boundary feedback and prove the exponential stability of the closed loop system. We show that for vanishing damping term in the partial differential equation, the decay rate of the system approaches the rate for the system governed by the wave equation without damping term. In particular, this implies that arbitrarily large decay rates can occur if the velocity damping in the partial differential equation is sufficiently small.

1. Introduction

We consider boundary feedback stabilization of a system governed by a semilinear wave equation with a velocity damping term. Special cases are also known as mildly damped wave equation or string with viscous damping or telegraph equation without leakage. The telegraph equation is used as a model for the voltage and current on a lossy transmission line. There is also a stochastic interpretation of the telegraph equation as the limit of a random migration, see [1,2]. In order to show that our approach also covers a nonlinear situation, we consider a semilinear system.

The exponential decay of the energy of the system for the case with velocity damping in the pde and homogeneous Dirichlet boundary conditions has been shown in [3].

Boundary feedback stabilization with memory type feedback has been studied in [4]. The boundary dissipation of memory type is discussed in [5]. In particular it includes as a special case velocity feedback boundary conditions where waves are reflected with a certain reflection coefficient. Velocity feedback for the wave equation has been analyzed for example in [6] where explicit decay rate estimates for the exponential decay are given. In particular in [6,7] it is stated that in the 1-d case, the energy decays to zero in finite time for a certain choice of the feedback parameter.

We are interested in the behavior of the decay rates for vanishing velocity term in the governing pde. In this case, the decay rate given in the main theorem in [4] converges to zero, that is the decay becomes arbitrarily slow. In this paper we show that in fact for the parameters given in [6], the decay rates become arbitrarily large with vanishing velocity damping term. This is for example interesting in the case that the velocity damping term occurs as a kind of perturbation or defect of the system. Our analysis shows that as a result of the interplay between the damping in the interior and the boundary damping, the decay rate approaches infinity if the damping in the interior goes to zero. This is the main result of this paper: The decay rate of the system with boundary damping and damping term in the pde approaches the decay rate of the system with boundary damping but without damping term in the pde as the damping term in the pde vanishes. In particular for the optimal feedback parameter if the interior damping is sufficiently small arbitrarily large decay rates can occur for the system with damping term in the pde as a result of the boundary damping.

Note that our result also applies in the case of anti-damping in the interior when the velocity term causes blow up in the case without boundary damping, for example in the linear case if the

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constant in the velocity term has the wrong sign for a damping term. Also in this case, the boundary damping generates exponential stability of the system if the velocity term in the pde is sufficiently small.

We start our analysis in an L^2-setting. Then we study solutions that are bounded almost everywhere, that is solutions that are defined in an L^∞-setting which is particularly attractive in the context of control. We also consider solutions in an H^1-framework that is often used for this type of system. For the construction of the solutions we use the method of characteristic curves. Our results are related to [8], where for a class of viscously damped vibrating systems it is shown that the energy of the system decays exponentially with a uniform rate for all viscous damping parameters in certain intervals. However, these results do not cover the case of boundary damping of the wave equation that we consider in this paper.

2. Definition of the system

Let initial data $y_0 \in L^2(0,1)$, $y_1 \in H^{-1} := \{y \in \mathcal{D}'(0,1) : y = Y'$ for some $Y \in L^2(0,1)\}$ be given. Let a twice continuously differentiable function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ be given, such that $g(x,0) = 0$ and there is a constant $w \geq 0$ such that for all $x \in (0,1)$ and all $y \in \mathbb{R}$ we have for the partial derivative g_y with respect to y

$$|g_y(x, y)| \leq w. \tag{1}$$

We consider the system with the initial conditions

$$y(0,x) = y_0(x), \qquad y_t(0,x) = y_1(x) \quad \text{for } x \in (0,1) \tag{2}$$

with the evolution in time that is governed by the semilinear pde

$$y_{tt}(t,x) - 2g_y(x, y(t,x))\, y_t(t,x) = y_{xx}(t,x) \tag{3}$$

and with the boundary conditions

$$y(t,0) = 0, \qquad y_x(t,1) = -f y_t(t,1) \quad \text{for } t > 0 \tag{4}$$

with a feedback parameter $f \in [0,\infty)$. In [9], we have shown that for $w = 0$, the choice $f = 1$ for the feedback parameter is optimal in well-defined sense. In this case with $f = 1$, the system comes to the zero position with zero velocity in finite time (see [6,7]).

As a special case, with $g(x,y) = -wy$ our equation has the form of the telegraph equation

$$y_{tt} + 2\,w\,y_t = y_{xx}. \tag{5}$$

3. Well-posedness

First we construct solutions of the system (2)–(4). We start with the construction of solutions that are for all $\bar{T} > 0$ in the function space $L^2((0,\bar{T}) \times (0,1))$. Then for more regular initial data, we define solutions that are in $L^\infty((0,\bar{T}) \times (0,1))$. Using these solutions, we finally show that for sufficiently regular initial data we obtain solutions in $L^\infty((0,\bar{T}) \times (0,1)) \cap H^1((0,\bar{T}) \times (0,1))$.

3.1. L^2-solutions

In Theorem 3 we will present solutions y that are for all $\bar{T} > 0$ in the function space $L^2((0,\bar{T}) \times (0,1))$. The first part of the representation of the solution defined below in (6) is $\alpha(x+t) + \beta(x-t)$, that is a traveling waves solution of the wave equation. The other parts are generated by the damping term in the pde. We are looking for a solution of the form

$$y(t,x) = \alpha(x+t) + \beta(x-t) + \gamma_+(t,x) + \gamma_-(t,x). \tag{6}$$

Here γ_+ and γ_- are defined as

$$\gamma_+(t,x) = \int_0^t g(k_+(s,x,t),\ y(s,\ k_+(s,x,t)))\, ds \tag{7}$$

$$\gamma_-(t,x) = \int_0^t g(k_-(s,x,t),\ y(s,\ k_-(s,x,t)))\, ds \tag{8}$$

thus $\gamma_+(0,x) = \gamma_-(0,x) = 0$ that is initially, γ_+ and γ_- vanish.

The map $k_-(\cdot,x,t)$ is the continuous time-periodic function with period 2 with values in $[0,1]$ that is uniquely defined by the equations

$$k_-(s,x,t) = x + t - s - 2n \tag{9}$$

if $x + t - s - 2n \in (0,1)$ for some $n \in \mathbb{Z}$ and

$$k_-(s,x,t) = s - x - t - 2n$$

if $s - x - t - 2n \in (0,1)$ for some $n \in \mathbb{Z}$.

Note that we can also consider $k_-(s,x,t)$ as a function of s and the sum $x + t$.

The function $k_+(\cdot,x,t)$ is the continuous time-periodic function with period 2 and values in $[0,1]$ that is uniquely determined by the equations

$$k_+(s,x,t) = x - t + s - 2n \tag{10}$$

if $x - t + s - 2n \in (0,1)$ for some $n \in \mathbb{Z}$ and

$$k_+(s,x,t) = t - x - s - 2n$$

if $t - x - s - 2n \in (0,1)$ for some $n \in \mathbb{Z}$. Note that $k_+(s,x,t)$ is a function of s and the difference $x - t$.

The curves $k_+(\cdot,x,t)$, $k_-(\cdot,x,t)$ are the characteristic curves of our system through the point (t,x) with the reflections at the boundaries $x = 0$ and $x = 1$. Note that k_+ and k_- are almost everywhere differentiable as piecewise linear functions.

The initial conditions (2) determine the values of $\alpha|_{(0,1)}$ and $\beta|_{(0,1)}$ as elements of $L^2(0,1)$. Similar as in the classical d'Alembert solution, for x almost everywhere in $(0,1)$ we have

$$\alpha(x) = \frac{1}{2}y_0(x) + \frac{1}{2}\int_0^x y_1(s) - 2g(s,\ y_0(s))\, ds + C_A, \tag{11}$$

$$\beta(x) = \frac{1}{2}y_0(x) - \frac{1}{2}\int_0^x y_1(s) - 2g(s,\ y_0(s))\, ds - C_A \tag{12}$$

with a real number C_A. Due to the regularities of y_0 and y_1, (11) and (12) imply that $\alpha, \beta \in L^2(0,1)$.

Note that $k_+(s,0,t) = k_-(s,0,t)$. By the definitions (7), (8), this implies

$$\gamma_-(t,0) = \gamma_+(t,0).$$

Now we want to derive equations that allow to compute the functions α and β recursively in such a way that the boundary conditions are satisfied. Representation (6) implies

$$y_x(t,x) = \alpha'(x+t) + \beta'(x-t) + \partial_x\gamma_+(t,x) + \partial_x\gamma_-(t,x), \tag{13}$$

$$y_t(t,x) = \alpha'(x+t) - \beta'(x-t) + \partial_t\gamma_+(t,x) + \partial_t\gamma_-(t,x). \tag{14}$$

Hence

$$\begin{aligned} y_x(t,x) + f y_t(t,x) = {} & (1+f)\alpha'(x+t) + (1-f)\beta'(x-t) \\ & + \partial_x\gamma_+(t,x) + \partial_x\gamma_-(t,x) + f\,\partial_t\gamma_+(t,x) \\ & + f\,\partial_t\gamma_-(t,x). \end{aligned}$$

Thus the boundary condition (4) at $x = 1$ implies for $t > 0$

$$\begin{aligned} \alpha'(1+t) = {} & -\frac{1}{f+1}\left[(1-f)\beta'(1-t) + \partial_x\gamma_+(t,1)\right. \\ & \left. +\ \partial_x\gamma_-(t,1) + f\,\partial_t\gamma_+(t,1) + f\,\partial_t\gamma_-(t,1)\right]. \end{aligned}$$