Systems & Control Letters 66 (2014) 94–103

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

On the consensus and bipartite consensus in high-order multi-agent dynamical systems with antagonistic interactions

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ARTICLE INFO

Article history:
Received 7 June 2013
Received in revised form
13 October 2013
Accepted 8 January 2014
Available online 25 February 2014

Keywords:
Consensus
Bipartite consensus
Multi-agents
Antagonistic interactions
Communication graph

ABSTRACT

The aim of this paper is to address consensus and bipartite consensus for a group of homogeneous agents, under the assumption that their mutual interactions can be described by a weighted, signed, connected and structurally balanced communication graph. This amounts to assuming that the agents can be split into two antagonistic groups such that interactions between agents belonging to the same group are cooperative, and hence represented by nonnegative weights, while interactions between agents belonging to opposite groups are antagonistic, and hence represented by nonpositive weights. In this framework, bipartite consensus can always be reached under the stabilizability assumption on the state-space model describing the dynamics of each agent. On the other hand, (nontrivial) standard consensus may be achieved only under very demanding requirements, both on the Laplacian associated with the communication graph and on the agents' description. In particular, consensus may be achieved only if there is a sort of "equilibrium" between the two groups, both in terms of cardinality and in terms of the weights of the "conflicting interactions" amongst agents.

1. Introduction

Mathematical formulation of multi-agents systems and consensus problems has been of interest for a considerable length of time. Some of the pioneering works are reported in [1–5] and references therein. However, a decade ago, thanks to milestone contributions such as [6–12], a wide stream of literature on these topics started and flourished. The driving force behind considerable research activity on this topic is the wide variety of areas where the consensus problem lends itself in a natural manner. In applications such as sensor networks, coordination of mobile robots or UAVs, flocking and swarming in animal groups, dynamics of opinion forming, etc., the problem can be formulated as that of a group of agents exchanging information with the objective of reaching a common decision, a consensus, by resorting to distributed algorithms that make use of the information that each agent collects from neighboring agents (see, e.g. [13–19,11,20,21]). The interested reader is referred to [22,10] for a more complete list of references.

A common assumption in most of the literature about consensus is that agents achieve this goal through collaboration. However, in a number of contexts where the consensus is meaningful, the interactions among agents are not necessarily cooperative. On the contrary, in contexts like markets or social networks [23], for instance, agents may also display non-cooperative or antagonistic interactions with some of their neighboring agents. In social networks, mutual relationships between pairs of individuals may be friendly or hostile, and this may create two antagonistic groups. In economic systems, duopolistic regimes arise quite frequently: all the companies producing a certain product, or providing a certain service, are split into two competing cartels. But this is also the case when modeling two competing teams, as it happens, for instance, in sport disciplines, or in robot competitions like RoboCup. Each individual or robot collects information regarding both the team mates and the antagonists, and processes this data in order to take decisions (position, speed, behavior, elevation, ...) that are in agreement with those of their team mates. Game theory provides several examples where players are divided into two competing teams, and antagonistic interactions between the two groups are crucial when modeling the overall system dynamics. Finally, in biological systems, interactions between genes or chemical elements may be cooperative or antagonistic in the form of activators/inhibitors.

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http://dx.doi.org/10.1016/j.sysconle.2014.01.006
0167-6911/

In graph theoretic terms, antagonistic interactions can be taken into account by replacing the standard communication graph, characterized by nonnegative weights, with a signed graph [24,25] displaying both positive and negative weights. Positive arcs correspond to cooperative interactions between agents, while negative arcs describe interactions between antagonistic agents. While there is considerable literature about consensus in cooperating multi-agent systems, research results pertaining to consensus without cooperation are relatively few [26–28]. In a recent paper [22], Altafini developed the concept of *bipartite consensus* among agents with antagonistic interactions. Specifically, based on definitions and properties of signed weighted (directed or undirected) graphs and of the associated Laplacians (see [24,29]), he introduced the concept of bipartite consensus (or agreed dissensus). This is the situation when agents are split into two groups such that within each group all the agents converge to a unique decision, but the decisions of the two groups are opposite.

By addressing the classical example of homogeneous agents modeled as simple scalar integrators, he proved that if the signed, weighted and connected communication graph describing the agents' interactions is structurally balanced, then the agents reach bipartite consensus. On the other hand, if the interactions are antagonistic, but not structurally balanced, the only agreement that can be achieved among the agents is the trivial one, where all the agents' states converge to zero.

The main objective of this paper is to extend the results reported in [22], by addressing the general case of N homogeneous agents described by a generic n-dimensional linear state-space model. This represents a natural extension of the case when agents are modeled as simple integrators. In certain situations, agents' status may require more than a single variable for accurate representation. These variables may include, e.g., position and velocity, price and production levels, etc. These decision variables are updated based on the information collected from the neighboring agents, and consensus must be achieved on all of them. Specifically, we establish conditions for consensus and bipartite consensus for a group of N homogeneous agents under the assumption that their mutual interactions can be described by a weighted, signed, connected and structurally balanced communication graph.

We show that, in this set-up, bipartite consensus can always be reached under the fairly weak assumption of stabilizability on the state-space model describing the dynamics of each agent. However, nontrivial consensus to a common decision for the two antagonistic groups can be achieved only under more restrictive requirements, both on the Laplacian associated with the communication graph and on the agents' description. In particular, consensus may be achieved only if there is a sort of "equilibrium" between the two groups, both in terms of cardinality and in terms of the weights of the "conflicting interactions" among agents.

Briefly, Section 2 introduces the problem formulation and formalizes the consensus and bipartite consensus problems. In addition, basic definitions and results regarding weighted signed graphs and their Laplacians are reviewed, and a new technical result regarding the Laplacian of structurally balanced graphs is presented. Section 3 investigates the bipartite consensus problem, and it is shown there that, under the structural balance assumption, it is possible to extend to the case of antagonistic interactions and bipartite consensus the results presented in [30,31] (see, also, [6,32,21,33]) for the consensus of high-order cooperating agents. Section 4 explores the consensus problem, by focusing on the case when the common trajectory that the agents converge to is bounded, but not converging to zero. In this section, conditions under which consensus may be achieved are investigated, and an algorithm to design the control law that makes this possible is presented. Finally, it is shown that when the previous conditions are not met, nontrivial consensus can never be achieved.

Notation. $\mathbb{R}_+$ is the semiring of nonnegative real numbers. For any pair of positive integers k and n with $k \leq n$, $[k, n]$ is the set of integers $\{k, k+1, \dots, n\}$. The (i, j)th entry of a matrix A will be denoted by $[A]_{ij}$, the ith entry of a vector $\mathbf{v}$ by $[\mathbf{v}]_i$. A matrix (in particular, a vector) A with entries in $\mathbb{R}_+$ is called *nonnegative*, and denoted by $A \geq 0$. The symbol $\mathbf{1}_N$ denotes the N-dimensional vector with all entries equal to 1. The *spectrum* of a square matrix A is denoted by $\sigma(A)$.

2. Consensus and bipartite consensus problems: statements

We consider a multi-agent system consisting of N agents, each of them described by the same single-input continuous-time state-space model. Specifically, $\mathbf{x}_i(t)$, the ith agent state, $i \in [1, N]$, evolves according to the first-order differential equation

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + bu_i(t), \tag{1}$$

where $\mathbf{x}_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, and $b \in \mathbb{R}^n$. The communication among the N agents is described by an *undirected, weighted and signed, communication graph* [22] $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs, and $\mathcal{A}$ is the matrix of the signed weights of $\mathcal{G}$. The (i, j)th entry of $\mathcal{A}$, $[\mathcal{A}]_{ij}$, is nonzero if and only if the arc (j, i) belongs to $\mathcal{E}$, namely information about the status of the jth agent is available to the ith agent. We assume that the interactions between pairs of agents are symmetric and hence $\mathcal{A} = \mathcal{A}^\top$. The interaction between the ith and the jth agents is cooperative if $[\mathcal{A}]_{ij} > 0$ and antagonistic if $[\mathcal{A}]_{ij} < 0$. Also, we assume that $[\mathcal{A}]_{ii} = 0$, for all $i \in [1, N]$. The graph $\mathcal{G}$ is *connected* if, for every pair of vertices j and i, there is path, namely an ordered sequence of arcs $(j, i_1), (i_1, i_2), \dots, (i_{k-1}, i_k), (i_k, i) \in \mathcal{E}$, connecting them.

The *Laplacian matrix* associated with the adjacency matrix $\mathcal{A}$ is defined as in [22,24,29], namely:

$$\mathcal{L} := \mathcal{C} - \mathcal{A}, \tag{2}$$

where $\mathcal{C}$ is the (diagonal) connectivity matrix, whose diagonal entries are the sums of the absolute values of the corresponding row entries of $\mathcal{A}$, namely

$$[\mathcal{C}]_{ii} = \sum_{(j,i)\in\mathcal{E}} |[\mathcal{A}]_{ij}|, \quad \forall i \in [1, N].$$

Therefore

$$[\mathcal{L}]_{ij} = \begin{cases} \sum_{(j,i)\in\mathcal{E}} |[\mathcal{A}]_{ij}|, & \text{if } i = j \\ -[\mathcal{A}]_{ij}, & \text{if } i \neq j. \end{cases} \tag{3}$$

Throughout the paper, we assume that the weighted and signed graph $\mathcal{G}$, describing the interactions among agents, is connected and *structurally balanced*. This latter property means [22,24,25] that the set of vertices $\mathcal{V}$ can be partitioned into two disjoint subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that for every $i, j \in \mathcal{V}_p$, $p \in [1, 2]$, $[\mathcal{A}]_{ij} \geq 0$, while for every $i \in \mathcal{V}_p, j \in \mathcal{V}_q$, $p, q \in [1, 2]$, $p \neq q$, $[\mathcal{A}]_{ij} \leq 0$. This amounts to saying that the agents can be split into two groups, and interactions between pairs of agents belonging to the same group are cooperative, while interactions between pairs of agents belonging to different groups are antagonistic. Therefore, after a suitable reordering of the agents, we can always assume that

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{12}^\top & \mathcal{A}_{22} \end{bmatrix}, \tag{4}$$

where $k := |\mathcal{V}_1|$, $N - k := |\mathcal{V}_2|$, $\mathcal{A}_{11} = \mathcal{A}_{11}^\top \in \mathbb{R}_+^{k\times k}$, $\mathcal{A}_{22} = \mathcal{A}_{22}^\top \in \mathbb{R}_+^{(N-k)\times(N-k)}$, while $-\mathcal{A}_{12} \in \mathbb{R}_+^{k\times(N-k)}$. Under this fundamental requirement on the mutual interactions among the agents, we