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# Analytical solutions for thermal vibration of nanobeams with elastic boundary conditions

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ABSTRACT

A nonlocal Euler beam model with second-order gradient of stress taken into consideration is used to study the thermal vibration of nanobeams with elastic boundary. An analytical solution is proposed to investigate the free vibration of nonlocal Euler beams subjected to axial thermal stress. The effects of the nonlocal parameter, thermal stress and stiffness of boundary constraint on the vibration behaviors of nanobeams are revealed. The results show that natural frequencies including the thermal stress are lower than those without the thermal stress when temperature rises. The boundary-constrained springs have significant effects on the vibration of nanobeams. In addition, numerical simulations also indicate the importance of small-scale effect on the vibration of nanobeams.



## 1. Introduction

Nanostructures have attracted considerable attention for their outstanding mechanical, chemical and thermal properties [1–4]. Hence, nanobeams hold exciting promise as transistors, probes, sensors, actuators, and resonators in nanoelectromechanical systems. Since the experiments are very difficult, and molecular dynamics simulations remain expensive at the nano-scale, continuum elastic models have been widely used to study wave propagation and vibration of nanobeams [5–9].

Experimental results and molecular dynamics simulation show that the size effects play a major role in the mechanical properties of microstructures [10–13] and nanostructures [5,6,14]. Because of the lacking of size-dependent material length-scale parameter, the classical continuum elasticity theory fails to describe the structural behavior at micron- and nanometer-scale accurately. To overcome this weakness, several nonlocal elasticity theories which incorporate the internal material length-scale parameters, such as the couple stress theory [15–17], the strain gradient theory [18–20] and the stress gradient theory [21–26], have been employed to describe the behaviors of material with microstructure and nanostructure. The stress gradient elasticity theory proposed by Eringen is a nonlocal model of the gradient type, which introduces high-order gradient of stress into the constitutive relation [21]. In recent years, many researchers have applied the stress gradient elasticity theory to the bending, buckling, and vibration analysis of nanostructures [22–26].

Thermal vibration analysis is needed for the nanobeams often subjected to thermal loading. Zhang et al. [27] developed a double-elastic beam model for studying transverse vibrations of double-walled carbon nanotubes. Benzair et al. [28] used the nonlocal Timoshenko beam model for free vibration analysis of single-walled carbon nanotubes including the thermal effect. Murmu and Pradhan [29] developed a single-elastic beam model to analyze the thermal vibration of single-walled carbon nanotubes based on nonlocal elasticity theory.

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Ebrahimi and Salari [30] investigated the thermal effect on the buckling and free vibration characteristics of the functionally graded size-dependent Timoshenko nanobeams subjected to an in-plane thermal loading.

In the above-mentioned studies, the boundary conditions of nanobeams are all classical cases, such as free, hinged or clamped ones. In practice, the interaction between nanobeams and substrates is van der Waals force. Thus the boundary condition of the nanobeam is commonly elastically constrained. It has been widely accepted that it is very hard to obtain an analytical solution for beams or plates except for very few simple boundary cases. Thus, some efficient numerical solution techniques, such as the modified Fourier series method, the meshless method and the differential quadrature method have been employed to solve the vibration problems of beams with arbitrary boundary conditions. Li and coworkers [31,32] and Jin et al. [33] presented a modified Fourier series method for the vibration of beams with general boundary conditions based on the classical Euler beam model. Kiani [34] used the reproducing kernel particle method to study the free transverse vibration of embedded single-walled nanotubes with arbitrary boundary conditions by the nonlocal Euler beam, Timoshenko beam, and higher-order beam models. Rosa and Lippiello [35] adopted the differential quadrature method to investigate free vibrations of embedded single-walled carbon nanotubes based on the Euler beam theory.

The attempt of this work is to propose an analytical solution for studying the vibration of elastically-supported nanobeams with second-order stress gradient elastic theory subjected to thermal stress. For this purpose, the nonlocal Euler beam model with elastic boundary is presented in Section 2. Then, an analytical solution for boundary value problems for the free vibrations of a nanobeam is derived in Section 3. Vibration analysis of the nanobeams with elastic boundary conditions are presented and discussed in Section 4. Finally, some concluding remarks are made in Section 5.

## 2. Nonlocal Euler beam model with elastic boundary

According to Eringen's nonlocal elasticity theory, the constitutive law between stress and strain in the one-dimensional case can be expressed as

$$\sigma_x - \mu^2 \frac{\partial^2 \sigma_x}{\partial x^2} = E\varepsilon_x, \tag{1}$$

where $E$ represents Young's modulus and $\varepsilon_x$ is the axial strain; $\mu = e_0 a$ is the nonlocal parameter reflects the influence of the microstructure on the strain in the nonlocal elastic material [21], with $e_0$ a constant appropriate to each material, and $a$ an internal characteristic length of the material.

For a thin beam, the displacement components can be written as

$$u_x = -z\frac{\partial w(x,y,t)}{\partial x},\ u_y = 0,\ u_z = w(x,y,t) \tag{2}$$

where $t$ denotes time, and $u$ and $w$ are displacements of the middle line. The strain field can be expressed as

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2}. \tag{3}$$

The relation between the shear force Q and the bending moment $M$ is expressed as

$$Q - \frac{\partial M}{\partial x} = 0, \tag{4}$$

with axial force considered,

$$\frac{\partial Q}{\partial x} = \rho A\frac{\partial^2 w}{\partial t^2} - N\frac{\partial^2 w}{\partial x^2}, \tag{5}$$

where $\rho$ is the mass density of the material, $A$ is the area of the cross section, and $N$ denotes an additional axial force dependent on temperature change $\Delta T$ and thermal coefficient $\alpha_T$ of the nanobeam, which can be expressed as

$$N = -EA\alpha_T \Delta T. \tag{6}$$

The bending moment

$$M = \int_A z\sigma_x dA. \tag{7}$$

A combination of Eqs. (1), (3) and (7) leads to

$$M - \mu^2\frac{\partial^2 M}{\partial x^2} = -EI\frac{\partial^2 w}{\partial x^2}, \tag{8}$$

where $I$ is the moment of inertia of the cross section.

Eliminating Q from Eqs. (4) and (5) yields

$$\frac{\partial^2 M}{\partial x^2} = \rho A\frac{\partial^2 w}{\partial t^2} - N\frac{\partial^2 w}{\partial x^2}. \tag{9}$$

Then, substituting Eq. (9) into Eq. (8), the bending moment $M$ for the nonlocal Euler beam model with axial force can be expressed as

$$M = -\mu^2\left(N\frac{\partial^2 w}{\partial x^2} - \rho A\frac{\partial^2 w}{\partial t^2}\right) - EI\frac{\partial^2 w}{\partial x^2}. \tag{10}$$

From Eqs. (4) and (10), the shear force Q for the nonlocal Euler beam model can be expressed as

$$Q = -\mu^2\left(N\frac{\partial^3 w}{\partial x^3} - \rho A\frac{\partial^3 w}{\partial t^2 \partial x}\right) - EI\frac{\partial^3 w}{\partial x^3}. \tag{11}$$

Substituting Eq. (11) into Eq. (5), the governing equation of the nonlocal elastic beam can be derived as

$$[EI + \mu^2 N]\frac{\partial^4 w}{\partial x^4} - N\frac{\partial^2 w}{\partial x^2} + \rho A\frac{\partial^2 w}{\partial t^2} - \mu^2\rho A\frac{\partial^4 w}{\partial x^2 \partial t^2} = 0. \tag{12}$$

In particular, the new beam model can be degenerated to the classical Euler beam model with axial force, if the material length-scale parameter is set to be zero.

The equivalent continuum model of a nanobeam supported by an elastic medium at both ends is shown in Fig. 1.