Acta Mechanica Solida Sinica 000 (2017) 1–7

Available online at www.sciencedirect.com
ScienceDirect
journal homepage: www.elsevier.com/locate/CAMSS


# Timoshenko inclusions in elastic bodies crossing an external boundary at zero angle

A.M. Khludnev[a,*], T.S. Popova[b]

[a] Lavrentyev Institute of Hydrodynamics of RAS, Novosibirsk State University, Novosibirsk 630090, Russia
[b] North-Eastern Federal University, Yakutsk 677000, Russia

ARTICLE INFO

*Article history:*
Received 3 February 2016
Revised 19 May 2017
Accepted 31 May 2017
Available online xxx

*MSC:*
35B30
74G65
74G55

*Keywords:*
Thin Timoshenko inclusion
Crack
Delamination
Fictitious domain method
Non-penetration

ABSTRACT

The paper concerns an analysis of equilibrium problems for 2D elastic bodies with a thin Timoshenko inclusion crossing an external boundary at zero angle. The inclusion is assumed to be delaminated, thus forming a crack between the inclusion and the body. We consider elastic inclusions as well as rigid inclusions. To prevent a mutual penetration between the crack faces, inequality type boundary conditions are imposed at the crack faces. Theorems of existence and uniqueness are established. Passages to limits are investigated as a rigidity parameter of the elastic inclusion going to infinity.



## 1. Introduction

A suitable description of composite materials requires to analyze high-level mathematical models of elastic bodies with thin inclusions and cracks. Inclusions may delaminate from the matrix, and thus introducing cracks. A free boundary approach is to be used to properly describe a mutual non-penetration between the crack faces.

The equilibrium state of composite materials strongly depends on the location of thin and volume inclusions. Also, the dependence of solutions on coefficients can be a complicated mathematical problem provided that the inclusions are delaminated. We should distinguish between thin elastic and rigid inclusions. On the one hand, we have to write equilibrium equations for the elastic inclusions by taking into account the forces from surrounding elastic media; on the other hand, we have to find a suitable displacement field for the rigid inclusions. The equilibrium state for a rigid inclusion is formulated in terms of nonlocal integral conditions. We consider a nonlinear system of inequality type at the crack faces to correctly describe the boundary conditions. In the case of rigid inclusions inserted in the elastic body, new types of nonlocal boundary conditions appear which lead to free

* Corresponding author.
*E-mail addresses:* khlud@hydro.nsc.ru (A.M. Khludnev), ptsokt@mail.ru (T.S. Popova).

http://dx.doi.org/10.1016/j.camss.2017.05.005
0894-9166/© 2017 Published by Elsevier Ltd on behalf of Chinese Society of Theoretical and Applied Mechanics.

ARTICLE IN PRESS
JID: CAMSS [m7;June 30, 2017;15:28]


boundary-value problems being corrected from the mechanical standpoint. We should remark at this point that the classical approach to the crack problems in elasticity theory is characterized by linear boundary conditions imposed at the crack faces [1–2]. Such a linear approach allows the opposite crack faces to penetrate each other, which leads to inconsistency from the practical standpoint. During the past years, the crack theory with non-penetration conditions has been under active study. The book [3] contains results for crack models with non-penetration conditions for a wide class of elasticity problems (see also [4–7]). As for other constitutive laws, readers are referred to [8]. Existence theorems and qualitative properties of solutions in equilibrium problems for elastic bodies with rigid inclusions can be found in [9–16]. We can also mention some papers discussing models of inclusions without delamination [17,18]. As for thin elastic inclusions with delamination, we refer to [19–22]. A number of shape control problems have been analyzed for elastic bodies with elastic and rigid inclusions [23–28]. Also, we can mention junction problems for different elastic objects, and in particular, for plates, beams, etc. [29–36]

In this paper, we consider a Timoshenko thin inclusion in the elastic body. The elastic structure analyzed in the paper, in fact, corresponds to a combination of the elastic Timoshenko beam and the elastic body. The inclusion is assumed to cross an external boundary of the body at zero angle. Moreover, it is assumed that the inclusion delaminates, and hence a crack appears between the inclusion and the elastic matrix. Nonlinear boundary conditions of inequality type are considered at the crack faces to exclude a mutual penetration between faces. Hence, the free boundary approach is used to describe an equilibrium state of the structure with boundary conditions of inequality type at the crack faces. A model of a thin rigid inclusion inserted in an elastic body is also analyzed in the paper. Different problem formulations are proposed for the models considered, relating to weak and strong solutions which are proved to be equivalent under suitable regularity conditions. We prove existence and uniqueness of the solutions and analyze limit cases describing a passage to infinity of the rigidity parameter associated with the elastic inclusion. The paper provides a rigorous analysis of equilibrium problems for elastic bodies with delaminated thin elastic and rigid inclusions crossing an external boundary at zero angle. Compared with the nonzero angle cases, the presence of zero angle between the inclusion and the external boundary provides an additional difficulty, because there is no Korn's inequality in this case. To overcome the difficulty, a fictitious domain method is elaborated. The solution existence of a family of suitable problems in the extended domain is first proved. A priori estimates uniform with respect to a fictitious domain parameter are then obtained, and a passage to a limit is justified as the parameter goes to zero.

We can compare the results of this paper with those of [37], where an Euler–Bernoulli inclusion crossing the external boundary of the elastic body is analyzed. It is seen that the limit models, as rigidity parameters tend to infinity, demonstrate a similarity. In fact, the basic difference is that the limit Timoshenko inclusion provides limiting rotation angle values for the inclusion.

The paper is organized as follows. We discuss the problem formulation for the elastic Timoshenko inclusion in Section 2.

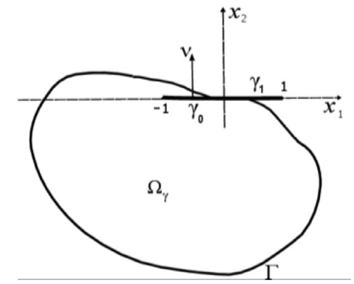


**Fig. 1 – Domain $\Omega_\gamma$ with thin inclusion $\gamma$.**

Section 3 is devoted to the analysis of solution existence for this problem. To this end, we propose a suitable fictitious domain method. In Section 4 an equilibrium problem with rigid inclusions is derived by treating the passages to limits as the rigidity parameter of the elastic inclusion that goes to infinity.

## 2. Elastic inclusion: setting of the problem

Let $\Omega \subset R^2$ be a bounded domain with Lipschitz boundary $\Gamma$, and $\gamma = (-1, 1) \times \{0\}$. Assume that $\gamma$ crosses $\Gamma$ at point (0, 0), and $\gamma_1 = (0, 1) \times \{0\} \not\subset \Omega$; $\gamma_0 = (-1, 0) \times \{0\} \subset \Omega$, as seen in Fig. 1. Denote $\boldsymbol{\nu} = (0, 1)$ as a unit vector normal to $\gamma$; $\boldsymbol{\tau} = (1, 0)$, and set $\Omega_\gamma = \Omega \backslash \bar{\gamma}$. We suppose that the angle between $\Gamma$ and $\gamma$ is zero at point (0, 0). A case of nonzero angle would be simpler, and suitable statements easily follow the results obtained below.

The domain $\Omega_\gamma$ represents a region filled with elastic material, and $\gamma$ is an elastic inclusion with specified properties. In particular, we consider $\gamma$ as a Timoshenko beam partially incorporated in the elastic body. Hence the beam is located inside the elastic body as well as outside it. Moreover, we assume that part of the beam, $\gamma_0$, delaminates, thus we have a crack between the beam and the elastic body. To provide a mutual non-penetration between crack faces, nonlinear boundary conditions will be considered at the faces.

By $\boldsymbol{A} = \{a_{ijkl}\}, i, j, k, l = 1, 2$, we denote a given elasticity tensor with the usual properties of symmetry and positive definiteness,

$$a_{ijkl} = a_{jikl} = a_{klij}, \quad i, j, k, l = 1, 2, \qquad a_{ijkl} \in L^{\infty}_{loc}(R^2)$$
$$a_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2 \quad \forall \xi_{ji} = \xi_{ij}, \quad c_0 = \text{const} > 0.$$

The summation convention over repeated indices is used; and all functions with two lower indices are assumed to be symmetric in those indices. Assume that $\boldsymbol{p} = (p_1, p_2, p_3) \in L^2(\gamma)^3$, $\boldsymbol{f} = (f_1, f_2) \in L^2_{loc}(R^2)^2$ are given functions acting on the inclusion $\gamma$ and the elastic body $\Omega_\gamma$, respectively.

An equilibrium problem for the body $\Omega_\gamma$ and the inclusion $\gamma$ is formulated as follows. We want to find the displacement field $\boldsymbol{u} = (u_1, u_2)$, and the stress tensor $\boldsymbol{\sigma} = \{\sigma_{ij}\}$, $i, j = 1, 2$, defined in $\Omega_\gamma$, as well as the thin inclusion displacements $w, v$ and the rotation angle $\varphi$ defined on $\gamma$ such that

$$-\text{div}\boldsymbol{\sigma} = \boldsymbol{f}, \quad \boldsymbol{\sigma} - A\varepsilon(\boldsymbol{u}) = \boldsymbol{0} \ \text{in} \Omega_\gamma \tag{1}$$

$$-\delta w_{,11} = [\sigma_\tau]\kappa_i + p_1, \quad -\delta v_{,11} - \delta\varphi_{,1} = [\sigma_\nu]\kappa_i + p_2 \ \text{on} \ \gamma_i, \quad i = 0, 1 \tag{2}$$

Please cite this article as: A.M. Khludnev, T.S. Popova, Timoshenko inclusions in elastic bodies crossing an external boundary at zero angle, Acta Mechanica Solida Sinica (2017), http://dx.doi.org/10.1016/j.camss.2017.05.005