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Technical note

On the wave dispersion and non-reciprocal power flow in space-time traveling acoustic metamaterials

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A B S T R A C T

This note analytically investigates non-reciprocal wave dispersion in locally resonant acoustic metamaterials. Dispersion relations associated with space-time varying modulations of inertial and stiffness parameters of the base material and the resonant components are derived. It is shown that the resultant dispersion bias onsets intriguing features culminating in a break-up of both acoustic and optic propagation modes and one-way local resonance band gaps. The derived band structures are validated using the full transient displacement response of a finite metamaterial. A mathematical framework is presented to characterize power flow in the modulated acoustic metamaterials to quantify energy transmission patterns associated with the non-reciprocal response. Since local resonance band gaps are size-independent and frequency tunable, the outcome enables the synthesis of a new class of sub-wavelength low-frequency one-way wave guides.

1. Introduction

The last few decades have witnessed a spurt of activity investigating the use of metamaterials to realize unique solutions to problems in vibroacoustic mitigation, wave cloaking, focusing, guidance, and others [1–3]. Locally resonant acoustic metamaterials (LRAMs) are sub-wavelength structures that exhibit mechanically tunable, size-independent, low frequency band gaps [4]. In their common form, LRAMs comprise a base (outer) structure that houses a series of uniformly distributed inner resonators (Fig. 1a), which contribute to the rise of unique dispersion properties. Local resonance band gaps in LRAMs stem from their ability to significantly attenuate incident excitations over a narrow frequency spectrum at the vicinity of the resonators' eigenfrequencies [5]. As such, LRAMs have been recently investigated in the context of discrete lumped mass systems [6,7], elastic bars [8,9], flexural beams [10–13], as well as 2D membranes and plates [14–16].

Owing to their periodic nature, band structures of LRAMs can be computed using a Bloch-Floquet wave solution. These structures convey the wave dispersion relations $\omega(\mu)$, where ω is the frequency and μ is the dimensionless wavenumber. Due to elastodynamic reciprocity, band structures of LRAMs are symmetric about $\mu = 0$ implying that waves travel from point A to B in the same manner they would travel from B to A [17]. Breaking this reciprocity in 1D systems creates a bias in the band structures intended to force waves to travel differently in opposing directions [18,19]. Non-reciprocity in metamaterials have been very recently utilized to synthesize, among others, acoustic guides [20] and static displacement amplifiers [21]. Means to induce non-reciprocal behavior include introduction of large nonlinearities, topological features, and material fields that travel in time and space [22,23]. The latter has been recently demonstrated in elastic metamaterials using a perturbation approach [24]. Although very challenging, several efforts have recently investigated achieving material variations in time using negative capacitance piezoelectric shunting [25], inductance-based resonance control [26], and magnetoelastic materials [27]. In this work, we build on the work developed in [28] for non-resonant space-time traveling phononic lattices to develop a mathematical framework that captures and predicts non-reciprocal dispersion physics in lumped time-traveling LRAMs. After analytically deriving the asymmetric wave dispersion relations based on a defined unit-cell, we validate the framework using the finite band structures reconstructed from the actual response of an LRAM chain of a known length. Furthermore, we present a structural intensity analysis of the non-reciprocal LRAMs and derive the power flow maps associated with the non-reciprocal energy transmission in the LRAM as a result of the imposed modulations.

This note is organized in four sections. Following the introduction, we begin by deriving the governing equations for spatiotemporally modulated mass and stiffness properties for both the base and the resonant components of a lumped LRAM to obtain the non-reciprocal dispersion relations. Through numerical simulations, a 2-Dimensional Fourier Transform (2D-FT) is then performed to validate the obtained band structure derived analytically. To further investigate the non-reciprocal behavior, in the subsequent section we investigate the LRAM

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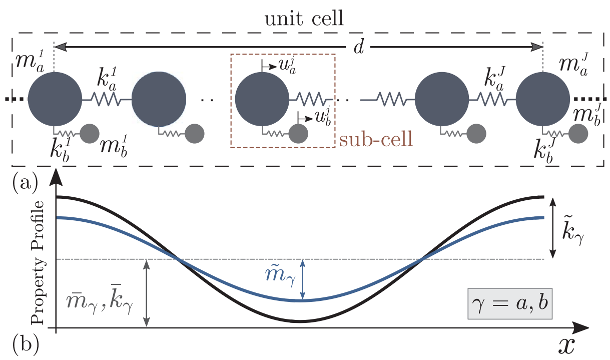

Fig. 1. (a) Lumped realization of a locally resonant acoustic metamaterial (LRAM) and (b) mass and stiffness modulation profile within a unit cell.

using the energy-based structural intensity analysis (SIA) to capture the power flow patterns within the non-reciprocal range. Finally, the conclusions are briefly summarized.

2. Dispersion relations

2.1. Mathematical formulation

To onset acoustic non-reciprocity, the parameters of the LRAM have to undergo a traveling-wave like modulation. As such, we begin by deriving mass m and stiffness k properties which travel simultaneously in time and space. Contrary to the conventional unit cell definition, we define a unit cell of a subset of lumped masses, spanning a length d, that constitute a single cycle of property variation (Fig. 1a). We also denote each spring-mass system and its resonator as a sub-cell. Consequently, we consider harmonic variation of m and k as follows

$$m_\gamma^j(t) = \overline{m}_\gamma + \widetilde{m}_\gamma \cos(\omega_0 t + \kappa_0 j) \tag{1}$$

$$k_\gamma^j(t) = \overline{k}_\gamma + \widetilde{k}_\gamma \cos(\omega_0 t + \kappa_0 j) \tag{2}$$

where, as depicted in Fig. 1b, $j = 1,\ldots,J$ is the sub-cell index and J is the total number of sub-cells within a unit cell. Also $\gamma = a,b$ refers to the base masses and local resonators, respectively. $\overline{k}_\gamma$ and $\overline{m}_\gamma$ are the average values of both variations while $\widetilde{k}_\gamma$ and $\widetilde{m}_\gamma$ are the oscillatory components. Further, ω_0 and $\kappa_0 = 2\pi/J$ represent the temporal and spatial modulation frequencies. In practice, such modulations can be physically realized via piezoelectric or magnetoelastic actuation [29]. Equations governing the motion of the jth sub-cell can be derived as

$$m_a^j \ddot{u}_a^j + (k_a^j + k_a^{j+1})u_a^j - k_a^j u_a^{j-1} - k_a^{j+1} u_a^{j+1} + k_b^j (u_a^j - u_b^j) = 0 \tag{3}$$

$$m_b^j \ddot{u}_b^j + k_b^j (u_b^j - u_a^j) = 0 \tag{4}$$

where u_a^j and u_b^j denote the base mass and resonator displacements, respectively. Using the Floquet-Bloch theorem [30,31], and exploiting the LRAM's periodicity, the unit cell displacement can be related to its adjacent ones via $u_\gamma^{J+1} = u_\gamma^1 e^{-i\mu}$ and $u_\gamma^0 = u_\gamma^J e^{i\mu}$, where i is the imaginary unit. Upon establishing periodic boundary conditions, the motion equations of the entire cell can be represented in compact matrix notation as

$$\mathbf{M}(t)\ddot{\mathbf{u}} + \mathbf{K}(t,\mu)\mathbf{u} = \mathbf{0} \tag{5}$$

where $\mathbf{u} = \{u_a^1, u_a^2, \ldots, u_a^J | u_b^1, u_b^2, \ldots, u_b^J\}^T$ is the displacement field, and $\mathbf{K}(t,\mu)$ and $\mathbf{M}(t)$ are the unit cell stiffness and mass matrices. Being periodic functions of time, both $\mathbf{K}$ and $\mathbf{M}$ can be expanded using a complex Fourier series as follows

$$\mathbf{M}(t) = \sum_{p=-\infty}^{\infty} \widehat{\mathbf{M}}_p e^{ip\omega_0 t} \tag{6}$$

$$\mathbf{K}(t,\mu) = \sum_{p=-\infty}^{\infty} \widehat{\mathbf{K}}_p e^{ip\omega_0 t} \tag{7}$$

where $\widehat{\mathbf{M}}_p$ and $\widehat{\mathbf{K}}_p$ are the corresponding Fourier matrix coefficients. Henceforth, we assume a harmonic solution with a time-modulated amplitude of the following form

$$\mathbf{u} = e^{i\omega t} \sum_{n=-\infty}^{\infty} \hat{\mathbf{u}}_n e^{in\omega_0 t} \tag{8}$$

where $\hat{\mathbf{u}}_n$ is the nth vector of displacement amplitudes. Substituting Eqs. (6)–(8) into (5), and employing harmonic function orthogonality, we obtain

$$\sum_{n=-N}^{N} (\mathbf{A}_{s,n}^{(2)}\omega^2 + \mathbf{A}_{s,n}^{(1)}\omega + \mathbf{A}_{s,n}^{(0)})\hat{\mathbf{u}}_n = \mathbf{0} \tag{9}$$

in which N is the truncated limit of the infinite series and s is an arbitrary integer within the interval $[-N,N]$. $\mathbf{A}_{s,n}^{(q)}$, with $q = 0,1,2$, is a $2J \times 2J$ matrix for any s and n combination. It is defined as

$$\mathbf{A}_{s,n}^{(0)} = n^2\omega_0^2 \widehat{\mathbf{M}}_{s-n} - \widehat{\mathbf{K}}_{s-n} \tag{10}$$

$$\mathbf{A}_{s,n}^{(1)} = 2n\omega_0 \widehat{\mathbf{M}}_{s-n} \tag{11}$$

$$\mathbf{A}_{s,n}^{(2)} = \widehat{\mathbf{M}}_{s-n} \tag{12}$$

Eqs. (9)–(12) can be combined into a quadratic eigenvalue problem [32]

$$(\mathbf{\Phi_2}\omega^2 + \mathbf{\Phi_1}\omega + \mathbf{\Phi_0})\hat{\mathbf{U}} = \mathbf{0} \tag{13}$$

where the new vector $\hat{\mathbf{U}}$ is obtained by stacking all $\hat{\mathbf{u}}_n$ for $n = -N$ to N, sequentially. The block matrix $\mathbf{\Phi_q}$ is of size $2J(2N+1) \times 2J(2N+1)$ and each of its elements is a sub-matrix defined by

$$\mathbf{\Phi_q}(s,n) = \mathbf{A}_{s,n}^{(q)} \qquad q = 0,1,2 \tag{14}$$

Eq. (13) requires the matrix multiplied by $\hat{\mathbf{U}}$ to be singular in order to yield a non-trivial solution, which describes the acoustic wave dispersion in the LRAM lattice. If an index p is defined such that $p = s-n$, explicit forms of $\widehat{\mathbf{M}}_p$ and $\widehat{\mathbf{K}}_p$, which constitute $\mathbf{A}_{s,n}^{(q)}$, can be found as

$$\widehat{\mathbf{M}}_p = \begin{bmatrix} \mathbf{M}_a^p & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_b^p \end{bmatrix} \tag{15}$$

$$\widehat{\mathbf{K}}_p = \begin{bmatrix} \mathbf{K}_a^p + \mathbf{K}_b^p & -\mathbf{K}_b^p \\ -\mathbf{K}_b^p & \mathbf{K}_b^p \end{bmatrix} \tag{16}$$

such that

$$\widehat{\mathbf{M}}_\gamma^p\left(r,j\right) = \left(\overline{m}_\gamma \delta_{p,0} + \frac{\widetilde{m}_\gamma}{2}\left(\delta_{p,-1}e^{i\kappa_0 j} + \delta_{p,1}e^{-i\kappa_0 j}\right)\right)\delta_{r,j} \tag{17}$$

$$\widehat{\mathbf{K}}_b^p\left(r,j\right) = \left(\overline{k}_b \delta_{p,0} + \frac{\widetilde{k}_b}{2}\left(\delta_{p,-1}e^{i\kappa_0 j} + \delta_{p,1}e^{-i\kappa_0 j}\right)\right)\delta_{r,j} \tag{18}$$

$$\widehat{\mathbf{K}}_a^p = \overline{k}_a \delta_{p,0}\mathbf{\Psi}_0 + \frac{\widetilde{k}_a}{2}(\delta_{p,-1}\mathbf{\Psi}_{-1} + \delta_{p,1}\mathbf{\Psi}_{+1}) \tag{19}$$

where $r = 1,\ldots,J$. The definitions of $\mathbf{\Psi_1}, \mathbf{\Psi_{-1}}$ and $\mathbf{\Psi_0}$ are given by

$$\mathbf{\Psi}_\ell = \begin{bmatrix} \psi_1^\ell + \psi_2^\ell & -\psi_2^\ell & & & -\psi_1^\ell e^{i\mu} \\ -\psi_2^\ell & \psi_2^\ell + \psi_3^\ell & -\psi_3^\ell & & \\ & \ddots & \ddots & \ddots & \\ & & & & -\psi_J^\ell \\ -\psi_1^\ell e^{-i\mu} & & & -\psi_J^\ell & \psi_J^\ell + \psi_1^\ell \end{bmatrix} \tag{20}$$

where $\psi_j^\ell = e^{-i\ell\kappa_0 j}$ and $\ell = -1,0,1$.