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Research paper

$W_{1+\infty}$ 3-algebra and the higher-order nonlinear Schrödinger equations in optical fiber

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ABSTRACT

In terms of $W_{1+\infty}$ 3-algebra, the generalized Nambu-Poisson evolution equations involving two Hamiltonians are constructed. Given the different Hamiltonian pairs, the higher-order nonlinear Schrödinger equations are obtained. Meanwhile, on the basis of the Maxwell equations, the higher-order nonlinear Schrödinger equations in optical fiber are derived in detail. Furthermore, the relations between the higher-order nonlinear Schrödinger equations in optical fiber and the higher-order nonlinear Schrödinger equations based on the $W_{1+\infty}$ 3-algebra are investigated. The bright soliton, dark soliton and the periodic traveling wave solutions of the higher-order nonlinear Schrödinger equation based on the $W_{1+\infty}$ 3-algebra are also derived.

1. Introduction

The $W_{1+\infty}$ algebra is an important infinite-dimension Lie algebra, which has wide applications in physics. It can be regarded as a linear deformation of the W_N algebras in the large N limit [1]. The W_N algebra is a higher-spin extension of the Virasoro algebra which has been found to be associated to the Korteweg-de Vries (KdV) equation [2,3]. 3-algebra is an elegant generalization of Lie algebra to the higher-order algebraic operations. It was introduced in the physics literature by Nambu and was developed in the mathematics literature by Filippov [4]. The use of 3-algebra is crucial to the construction. 3-algebra is the trilinear operations performed either on three operators or on three functions defined on a manifold. Recently 3-algebra has been applied to multiple M2-branes and the string/M-theory [5,6]. The properties of 3-algebras, in particular the infinite dimensional 3-algebras, have been widely investigated, such as (q-deformed) Virasoro-Witt 3-algebra [7,8], Kac-Moody 3-algebra and (super) w_∞ 3-algebra [9,10].

More recently, the relation between the infinite-dimensional 3-algebras and the integrable systems has been investigated [11–13]. The relation between the $W_{1+\infty}$ 3-algebra and the KP hierarchy has been constructed [14]. The KP equation is a higher-dimension generalization of the KdV equation. The KdV equation is a mathematical model of waves on shallow water surfaces. The first Hamiltonian structure of the KP equation was shown to be related to the $W_{1+\infty}$ algebra [15,16]. In theoretical physics, the nonlinear Schrödinger (NLS) equation is a classical field equation. One of its principal applications

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is the propagation of the wave in nonlinear optical fibers [17,18]. The equation models many nonlinear effects in a fiber, including the self-phase modulation, the group velocity dispersion, self-steepening and stimulated Raman scattering, etc. Additionally, the NLS equation appears in the studies of Bose-Einstein condensates [19–21], plasma physics [22,23], hydrodynamics [24], and so on. The relation between the classical NLS equation and the KP hierarchy has been established [25]. So the NLS equation is related to the $W_{1+\infty}$ algebra.

In reference [14], we also investigated the relation between the $W_{1+\infty}$ 3-algebra and the generalized NLS hierarchy, and derived some integrable and non integrable equations. As we know, many equations in physical applications are not integrable and the integrable equation is just an ideal equation. Thus we try to find the physical applications of these non integrable equations. In optical fiber, we caught sight of the generalized NLS equation with three-order dispersion effect. This equation is not integrable, but it is widely applied in optical fiber. In particular, more and more people pay attention to the application of the higher-order NLS equation, as the higher-order terms means the nonlinear effects. This urges us to explore the relation between the generalized higher-order NLS equations based on the $W_{1+\infty}$ 3-algebra and the higher-order NLS equations in optical fiber.

In this paper, we shall derive a family of higher-order NLS equations based on the $W_{1+\infty}$ 3-algebra. These equations are expected to be further applied in optical fiber. This paper is arranged as follows. In Section 2, based on some linear combination, we investigate the generalized Nambu-Poisson evolution equation based on the $W_{1+\infty}$ 3-algebra, and obtain the higher-order NLS equations. In Section 3, starting from the Maxwell equations, we derive the higher-order NLS equations in optical fiber. In Section 4, we compare the higher-order NLS equations in two forms, and find that the higher-order NLS equations in optical fiber can become the equations based on the $W_{1+\infty}$ 3-algebra under some transformations. We also derive the solitary wave solutions of the higher-order NLS equations. In Section 5, we present a short summary and further discussion.

2. The NLS equations based on $W_{1+\infty}$ 3-algebra

The operator Nambu 3-bracket was defined to be a sum of commutators of two operators multiplying the remaining one [26], i.e.,

$$[A, B, C] = [A, B]C + [B, C]A + [C, A]B, \tag{1}$$

where $[A, B] = AB - BA$.

Let us take $\{W_m^r = (-1)^r e^{\mathbf{i}mx} \frac{d^r}{dx^r} | m, r \in \mathbb{Z}, r \geq 0, \mathbf{i} = \sqrt{-1}\}$. Based on the operator Nambu 3-bracket (1), after a straightforward calculation we obtain the $W_{1+\infty}$ 3-algebra

$$\begin{aligned}
[W_m^r, W_n^s, W_k^h] = & \Big(\sum_{p=0}^{r}(-\mathbf{i})^p C_r^p n^p - \sum_{p=0}^{s}(-\mathbf{i})^p C_s^p m^p\Big) \sum_{q=0}^{r+s-p} (-\mathbf{i})^q C_{r+s-p}^q k^q W_{m+n+k}^{r+s+h-p-q} \\
& + \Big(\sum_{p=0}^{s}(-\mathbf{i})^p C_s^p k^p - \sum_{p=0}^{h}(-\mathbf{i})^p C_h^p n^p\Big) \sum_{q=0}^{s+h-p} (-\mathbf{i})^q C_{s+h-p}^q m^q W_{m+n+k}^{r+s+h-p-q} \\
& + \Big(\sum_{p=0}^{h}(-\mathbf{i})^p C_h^p m^p - \sum_{p=0}^{r}(-\mathbf{i})^p C_r^p k^p\Big) \sum_{q=0}^{h+r-p} (-\mathbf{i})^q C_{h+r-p}^q n^q W_{m+n+k}^{r+s+h-p-q}.
\end{aligned} \tag{2}$$

Introduce the FTF as follows:

$$v_r(x) = \frac{1+\mathbf{i}}{2\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} W_m^r e^{-\mathbf{i}mx}, \tag{3}$$

and define the classical Nambu bracket $\{\ ,\ \} = -\mathbf{i}[\ ,\ ]$. Then we can rewrite the $W_{1+\infty}$ 3-algebra (2) as the following Nambu 3-bracket relation:

$$\begin{aligned}
&\{v_r(x), v_s(y), v_h(z)\} \\
&= \sum_{p=0}^{r}\sum_{q=0}^{r+s-p} (-1)^{p+q} C_r^p C_{r+s-p}^q v_{r+s+h-p-q}(x)\partial_x^p \delta(x-y)\partial_x^q \delta(x-z) \\
&- \sum_{p=0}^{s}\sum_{q=0}^{r+s-p} (-1)^{p+q} C_s^p C_{r+s-p}^q v_{r+s+h-p-q}(y)\partial_y^p \delta(y-x)\partial_y^q \delta(y-z) \\
&+ \sum_{p=0}^{s}\sum_{q=0}^{s+h-p} (-1)^{p+q} C_s^p C_{s+h-p}^q v_{r+s+h-p-q}(y)\partial_y^p \delta(y-z)\partial_y^q \delta(y-x) \\
&- \sum_{p=0}^{h}\sum_{q=0}^{s+h-p} (-1)^{p+q} C_h^p C_{s+h-p}^q v_{r+s+h-p-q}(z)\partial_z^p \delta(z-y)\partial_z^q \delta(z-x)
\end{aligned}$$