Commun Nonlinear Sci Numer Simulat 64 (2018) 135–148

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Extremely low order time-fractional differential equation and application in combustion process

Qinwu Xu[a], Yufeng Xu[b,*]

[a] Department of Mathematics, Nanjing University, Jiangsu, Nanjing 210093, PR China
[b] Department of Applied Mathematics, Central South University, Hunan, Changsha 410083, PR China

ARTICLE INFO

Article history:
Received 18 December 2017
Revised 22 April 2018
Accepted 24 April 2018
Available online 25 April 2018

MSC:
34A08
35B44
80A25
65M06

Keywords:
Fractional reaction–diffusion equation
Blow-up phenomenon
Adaptive finite difference method
Combustion process

ABSTRACT

Fractional blow-up model, especially which is of very low order of fractional derivative, plays a significant role in combustion process. The order of time-fractional derivative in diffusion model essentially distinguishes the super-diffusion and sub-diffusion processes when it is relatively high or low accordingly. In this paper, the blow-up phenomenon and condition of its appearance are theoretically proved. The blow-up moment is estimated by using differential inequalities. To numerically study the behavior around blow-up point, a mixed numerical method based on adaptive finite difference on temporal direction and highly effective discontinuous Galerkin method on spatial direction is proposed. The time of blow-up is calculated accurately. In simulation, we analyze the dynamics of fractional blow-up model under different orders of fractional derivative. It is found that the lower the order, the earlier the blow-up comes, by fixing the other parameters in the model. Our results confirm the physical truth that a combustor for explosion cannot be too small.

1. Introduction

Fractional derivative means the order of differentiation is not necessary an integer. Nowadays it has gained considerable development, and has been applied in many scientific as well as engineering fields: diffusion process, solid mechanics, economics and finance modelling, and so on. For more details, we refer to [1–4]. Specially, differential equations with fractional derivatives are extensively applied in modelling anomalous diffusion phenomenon, which leads to the so-called fractional diffusion equations. By extending classic Fick's law and Lévy flight to more general forms, one can obtain temporal fractional diffusion equation and spatial fractional diffusion equation respectively [5]. Alternatively, they can also be generated by directly replacing the integer-order partial derivatives in classic diffusion models with certain classes of fractional derivatives. For example, among temporal fractional diffusion equations, the order of fractional derivative acting on temporal variable distinguishes the characteristics of diffusion as diffusion-wave (the order lies between one and two), and subdiffusion (the order lies between zero and one). The subdiffusion phenomenon can be frequently found in many anomalous diffusion problem which has memory property, e.g., see [6,7]. Its heavy tail feature is considered much better than exponential decay in modelling many chemical and physical processes, such as heat conduction.

* Corresponding author.
E-mail addresses: xuqinwu@nju.edu.cn (Q. Xu), xuyufeng@csu.edu.cn (Y. Xu).

https://doi.org/10.1016/j.cnsns.2018.04.021

It seems that combustion is another topic which does not relate to fractional calculus [8,9]. Combustion process is usually understood as a high-temperature exothermic redox chemical reaction between a fuel (the reductant) and an oxidant, such as coal burning. However, it covers more similar phenomena including metal materials become oxidized when exposed in air. The mathematical model for description of the rapid exothermic reaction in combustion is generally of highly nonlinear, extremely complex, degenerate, quasilinear parabolic partial differential equations (see [10–14] and references therein). Whenever parameters in these models exceed their thresholds, the models develop local existence, multiple singularities or discontinuity, such as appearance of quenching or blow-up phenomena in a finite time. Moreover, blow-up phenomenon is significant for its important physical and engineering interpretations, as well as special applications in the manufacturing industry. It serves as an indicator in rigid ignition models of combustion process, e.g., burning and explosion of fuel in the internal combustion engine. Recently, fractional derivatives modelling has been applied to combustion process and more interesting features and dynamics of models are discovered [15,16]. Nevertheless, the investigations for blow-up phenomenon of fractional differential equation and establishing theoretical criteria for its appearance are far from their ends. This encourages us to study the blow-up phenomenon for some slow oxidized processes in combustion by using analogical models with some necessary modifications. In consideration of laws of conservation of mass, species, momentum and energy are employed to construct the mentioned models, and these laws now have been extensively generalized by using fractional calculus and nonlocal theory [17–19], we shall reconsider combustion process by using parabolic equations with fractional derivative.

The basic motivation of this paper is three-fold. First of all, fractional differential equation with extremely low order of derivative is not extensively studied yet. Theoretically, their mathematical properties and potential applications are not clear. In numerous published works, particularly in numerical studies, the equations and models with order of derivative lying between one-half and one (or greater than one) are frequently considered. Besides, the kernel function in Riemann–Liouville and Caputo derivatives (the definitions can be found in [2]) is weakly singular when order is in between one-half and one, while it becomes strongly singular if the order is smaller than one-half and very close to zero. Moreover, when the order of temporal fractional derivative goes to zero, the considered fractional diffusion equation reduces to a stationary equation where only Laplacian operator is left. The global and local features of solutions of the resulting steady-state equation connecting to the original fractional diffusion model are also not clear. However, it has been shown that the speed of blow-up significantly depends on the order of derivative [20,21]. Thus as it approaches to zero, the fractional diffusion equations can be used to model more complicated reaction phenomenon in combustion problem.

Secondly, since the kernel is more singular with extremely low order, numerical approximation is very sensitive with numerical method. Explicit discretization methods fail to converge as derivative order close to zero due to the harsh limitation on time steps [22]. The so-called L1-type finite difference scheme with uniform mesh is of convergent order close to two. However, this is true only for the area smooth enough in the interior part of time domain. In the initial moments, solution is nearly singular for equation with very low order fractional derivative, and numerical approximation becomes terrible. Even more, for problems with blow-up solution, gradient of solution tends to infinity when time goes close to blow-up time. Time discretization with extreme small step length is needed to obtain a reliable solution. Hence, an efficient adaptive time discretization is in urgent need. Adaptive step size methods have been investigated in many works, including adaptive strategies for both classical and fractional differential equations [23–25]. The main idea of most adaptive strategies is to find an optimal deformation map between equidistant grids and non-equidistant grids. This idea will also be the starting point of our method. For spatial discretization, a flexible high-order discretization will help to reduce computational cost and catch the large gradient change of the solution. Obviously, discontinuous Galerkin (DG) method is a good choice for our problem. The DG method first was introduced by Reed and Hill for solving the neutron transport equation [26]. Since they use completely discontinuous functions as basis functions, they have the flexibility which is not shared by typical finite element methods [27]. Now, DG methods have been studied thoroughly from both theoretical aspects and applications by many scholars [28–31]. With increasing attentions are paid to fractional problems in recent years, application of DG methods to fractional problems also has gained rapid development. Different versions of DG methods have been developed for temporal or spatial fractional differential equations [32–37].

Finally, combustion process has been familiarly modeled by conventional reaction–diffusion equations [8,9]. Nevertheless, there are very few efforts on the application of fractional differential equation with particular low order of fractional derivative in this field. The combustion process involves the phenomenon of rapid chemical reaction, such as burning, and also contains the slow oxidation process, for instance, under natural conditions, the oxidized process of copper is very slow and uncertain. To model the latter case, according to the ability of fractional subdiffusion equation, it can be applied to describe the mentioned physical process and chemical reaction.

Based on the discussion above, in this paper, we shall study the following time-fractional ignition model in combustion process

$$\begin{cases} \partial_{0+}^{\alpha} u = \partial_{xx} u + \delta f(u), & (x,t) \in \Omega := [0,L] \times [0,T), \\ u = u_0(x), & (x,t) \in [0,L] \times \{t=0\}, \\ u = 0, & (x,t) \in \{x = 0, L\} \times [0,T), \end{cases} \tag{1.1}$$

where $0 < \alpha \le 1$ and α might be very close to zero, $\delta \ge 0$ is a parameter, and $f(u)$ is a nonlinear source term representing the input energy. In some concrete cases, f may explicitly relate to the temperature u, location x and time t. $\partial_{0+}^{\alpha} u$ is the αth