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Research paper

The exact measures of the Sierpiński d-dimensional tetrahedron in connection with a Diophantine nonlinear system

Fabio Caldarola

Department of Mathematics and Computer Science, Cubo 31/B, University of Calabria, Rende, CS 87036, Italy

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ABSTRACT

The Sierpiński d-dimensional tetrahedron Δ^d is the generalization of the most known Sierpiński gasket which appears in many fields of mathematics. Considering the sequences of polytopes $\{\Delta_n^d\}_n$ that generate Δ^d, we find closed formulas for the sum $v_n^{d,k}$ of the measures of the k-dimensional elements of Δ_n^d, deducing the behavior of the sequences $\{v_n^{d,k}\}_n$.

It becomes quite clear that traditional analysis does not have the adequate language and notations to go further, in an easy and manageable way, in the study of the previous sequences and their limit values; contrariwise, by adopting the new computational system for infinities and infinitesimals developed by Y.D. Sergeyev, we achieve precise evaluations for every k-dimensional measure related to each Δ^d, obtaining a set $W = \{v_{①}^{d,k}\}_{d,k}$ of values expressed in the new system, which leads us to a Diophantine problem in terms of classical number theory.

To solve it, we work with traditional tools from algebra and mathematical analysis. In particular, we define two kinds of equivalence relations on W and we get a detailed description of the partition of various of its subsets together with the exact composition of the corresponding classes of equivalence.

Finally, we also show as the unique Sierpiński tetrahedron for each dimension d, is replaced, if we adopt Sergeyev's framework, by a whole family of infinitely many Sierpiński d-dimensional tetrahedrons.

1. Introduction

The Sierpiński gasket, also called the Sierpiński triangle, is one of the most known and very popular fractal set; it has the exterior shape of an equilateral triangle, and is subdivided recursively into smaller similar triangles, from which the central ones are removed at each step. It has been originally constructed as a curve by the Polish mathematician Wacław Sierpiński one hundred years ago [27,28], but it appeared as a decorative pattern many centuries before, for example in Italian medieval art (as in the Cosmati mosaics, see [30, pp. 43, 873]) and in particular, in several Roman churches and Basiliche from the 11th century (see [7]).

From the original work of Sierpiński, what is known as the *Sierpiński arrowhead curve* is a continuous map of the line segment $[0, 1] \subset \mathbb{R}$ whose image is a fractal curve identical to the Sierpiński gasket; but there are dozens of other different

E-mail address: caldarola@mat.unical.it

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ways to build the Sierpiński gasket and dozens of contexts, in mathematics and other disciplines, where it comes out and finds a variety of applications. For example, it arises from cellular automata (elementary cellular automaton rule 60, 90, 102 and many similar others; see [30] for their extensive descriptions), from chaos games and chaos theory, from puzzle graphs [26], from Pascal's triangle, etc., and it has many uses and applications ranging from engineering and technology like fractal antennas (see [2]), to programming and computer science, and also, for instance, even in music [13].

In this paper we study the generalization of the Sierpiński triangle in each dimension $d \geq 2$: the construction of this fractal is well known for every d (for a comprehensive introduction see for example [20], whose first part is entirely devoted to such fractal in dimension 2, 3 and Section 3.8 to general d, or the paper [24] for connections with matrices, digits and "inner products") together with some of its main characteristics as the fractal dimension (see [19]). Although different sources deal with the general d-dimensional case, they are, however, very few in comparison with all the literature on the cases with $d = 2$ and $d = 3$: for instance, the formula obtained in Proposition 2.1, which represents the starting point for the investigations of this paper, seems itself not known before.

The core of the article is, instead, the emergence of a Diophantine nonlinear system with a rather complicated formulation (see (24)) from an unsuspected context as d-dimensional fractal geometry: it is interesting *per se* for number theory, but represents much more than a Diophantine problem because it derives from the application to such fractal of a new computational methodology, recently introduced by Y.D. Sergeyev, which throws new light on the subject. In particular, in contexts similar to ours, this method allow us to consider k-dimensional elements of a fractal object and to determine their *exact measures* as it was an ordinary d-polytope; more generally instead, such a new system allows one to work *numerically* with infinities and infinitesimals in a handy way and, as it is easy to imagine, it is particularly useful in relation to the behavior of models or objects when they are viewed at "infinity".

For detailed introduction surveys on this new numerical system, the reader can see [33,37,42,43] and the book [31] written in a popular way. We inform that this computational methodology has already been successfully applied in optimization and numerical differentiation (see [11,12,38,49]) and in a number of other theoretical and computational research areas such as cellular automata (see [9,10] and in the context of [3] under investigation), Euclidean and hyperbolic geometry (see [21,22]), percolation (see [17,18,29]), fractals (see [5,29,32,34,40,44]), the Riemann zeta function, infinite series and Z-transform (see [6,35,39,48]), the first Hilbert problem, Turing machines and supertasks (see [25,36,45,46]), numerical solution of ordinary differential equations (see [1,23,41,47]), etc.

As regards the structure of the present article and some details on its content, in Section 2 we introduce the Sierpiński tetrahedron Δ^d in any dimension $d \geq 2$ together with its generating sequence of d-polytopes $\left\{\Delta_n^d\right\}_n$. Then we found, by the above mentioned Proposition 2.1, closed formulas depending on n, for the sum of the measures of the k-dimensional elements of Δ_n^d, and this gives the starting point of the research in this paper. First we deduce the limit measure properties of Δ^d by using classical analysis: this means that, when d and k vary, we obtain a family of elements denoted by $\left\{v_\infty^{d,k}\right\}$, but whose value is zero or $+\infty$ in almost all the cases. Despite they arise from different kinds of measures, in different dimensions, and they carry different meanings and contents, when two of them are both zero or both $+\infty$, they are clearly indistinguishable using the notations of traditional analysis.

In Section 3 instead, we consider the correspondent values of $\left\{v_\infty^{d,k}\right\}$ but using the new computational system: they are denoted by $v_{①}^{d,k}$ and their analysis opens new features and appears from the beginning, full of meaning and rich in interpretations. In addition, contrary to what happens to the values of the family $\left\{v_\infty^{d,k}\right\}$, the elements $v_{①}^{d,k}$ are very often quite different from each other as we will realize in a few simpler cases from (9) to (10), but it is not trivial to say to what point they are really different.

It is also quite evident that some pairs of infinities or infinitesimals arising from (9) to (10), are much more similar than others and this makes us want a sort of classification that emphasizes affinities and relationships. For this purpose, then, we define two equivalence relations between the elements $v_{①}^{d,k}$ (and in general among the numbers of the new computational system, see Definition 3.1), one stronger that the other, and we give by Theorem 3.1, the partition into equivalence classes of the set of the elements $v_{①}^{d,k}$, together with complete information as the number of classes, a set of "minimal" representatives, etc. In particular, in Theorem 3.1 (ii) we obtain that the equivalence classes relative to the stronger relation are all constituted of a single element, and, as consequence, we conclude that the elements of the new system $v_{①}^{d,k}$ are all distinct unlike the traditional case (see Corollary 3.1). Another consequence is that the Diophantine nonlinear system (24) we mentioned earlier, has no nontrivial integer solutions (Corollary 3.2).

Section 4 changes the way of arguing and shows as the unique Sierpiński tetrahedron in dimension d, is replaced, if we use the new computational system, by a whole family of infinitely many Sierpiński d-dimensional tetrahedrons, that give rise to a still larger family of related values $v_{r,①}^{d,k}$. We end the section by suggesting some further directions of research and how it is possible to generalize the results obtained in the previous sections.

Finally, Section 5 is devoted to the conclusions.

Regarding the notations, we advice that, as usual, we will write indifferently $\{a_n\}_n$, $\{a_n\}$, or sometimes simply a_n, to denote a sequence. Moreover, the symbol $\mathbb{N}$ denotes the set of positive integers for us, whilst $\mathbb{N}_0$ includes also zero.