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Research paper

# Analysis of noise effects in a map-based neuron model with Canard-type quasiperiodic oscillations

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A problem of the mathematical modeling and analysis of the complex mixed-mode stochastic oscillations in neural activity is studied. For the description of noise-induced transitions between regimes of neuron dynamics, 2D map-based system near Neimark–Sacker bifurcation is used as a conceptual model. We focus on the parametric zone of Canard explosion where the attractors (closed invariant curves) are extremely sensitive to noise. Using direct numerical simulation and semi-analytical approach based on the stochastic sensitivity analysis, we study the noise-induced transformations from unimodal oscillations to bimodal spiking oscillations. The supersensitive invariant curve which marks the epicenter of the Canard explosion is found. It is shown that for this curve, the noise-induced splitting occurs for extremely small random forcing. Changes of the amplitude and frequency properties of the stochastic mixed-mode oscillations are studied. The phenomenon of noise-induced transitions from order to chaos is discussed.



## 1. Introduction

Mathematical modeling and analysis of the neural activity is one of the intensively developing branches of nonlinear science [1–3]. A clarification of the fundamental mechanisms of mutual transitions between quiescence and excitation requires the use the dynamic models with strong nonlinearities and complex bifurcations. Studies of the behavior of neural networks are based on the numerical simulations of systems with a very large number of interconnecting neurons. In these circumstances, it is extremely important to use simple but quite representative models that can replicate the variety of the dynamics of the individual neuron. Here, the phenomenological map-based models attract attention of researchers in studies both networks and individual neurons [4–8].

In modeling complex neural networks, the system proposed by Rulkov is widely used [9–13]. This two-dimensional map mimics the main features of the biological neuron behavior, such as the quiescence, tonic spiking and bursting.

A specific feature of the excitable neuron is the fast transition from one regime to another under the insignificant stimulus change. For the modeling such excitable neurons, the dynamic systems with the Canard explosion are the most appropriate. In parametric zones close to Canard explosion, even small deterministic or stochastic disturbances result in the drastic changes in dynamics with the generation of the multi-modal oscillations and transitions to chaos. In the class of continuous-time neural systems, Canard-type cycles were studied in FitzHugh–Nagumo, Morris–Lecar, Hindmarsh–Rose,

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Hodgkin–Huxley models [14–18]. More complicated multi-modal regimes of excitement connected with the Canard-type tori have been discovered in the model of the Purkinje cell and in the 3D Hindmarsh–Rose model [19–24].

The phenomenon of Canard explosion in the discrete neuron models with piecewise discontinuous maps was studied in [25,26] where the detailed bifurcation analysis of the deterministic model was presented and the stochastic generation of the spiking was discovered in the zone of deterministic small-amplitude almost sinusoidal oscillations. In the present paper, we focus on the study of probabilistic mechanisms of the stochastic excitement in 2D map-based neuron models with Canards.

A wide diversity of stochastic phenomena in map-based dynamic systems is actively studied by researchers from various domains of nonlinear science (see, e.g.[27–29]). However, until now the main tool of the research is the time-consuming direct numerical simulation of the random states. A full mathematical description of the stochastic dynamics in discrete-time systems can be found from the functional Perron–Frobenius equation [29,30]. However, an analytical solution of such equations is known only for very specific examples. In [31,32], the constructive semi-analytical method for the approximation of the dispersion of random states around deterministic attractors has been proposed. A main idea of this method is connected with the description of the stochastic sensitivity of these attractors. This stochastic sensitivity function (SSF) technique allows us to analyze noise-induced phenomena in both regular and chaotic systems [33–35]. In the present paper, we apply this technique to the analysis of noise-induced mixed-mode oscillations in the zone of Canard explosion in map-based models. As a basic system, we consider the conceptual two-dimensional Rulkov neuron model [9] defined by the smooth map.

In Section 2, we consider the deterministic Rulkov model in the parametric zone with the Neimark–Sacker bifurcation. As a result of this bifurcation, the equilibrium loses its stability, and the stable closed invariant curve (CIC) appears. In the present paper, we focus on the parametric sub-interval where the system undergoes the Canard explosion. Specific features of Canard-type CICs are analysed by Lyapunov exponents and rotation number.

In Section 3, we consider how random noise impacts on the closed invariant curves. Here, we study the dispersion of random states in the stochastically forced CICs for the parameters close to the epicenter of the Canard explosion. Here, the supersensitive CIC is found.

The semi-analytical approach to the parametric analysis of the stochastic peculiarities of Canard-type CICs is used in Section 4. The SSF technique allows us to find the quantitative description of the randomly forced CICs. Here, we show how the spatial dispersion of random states around deterministic CICs can be effectively estimated by confidence bands.

In Section 5, amplitude and frequency characteristics of the noise-induced mixed-mode oscillations are studied. We show how increasing noise transforms unimodal oscillations into bimodal spiking oscillations. Here, mean values of interspike intervals are analysed. In Section 6, we consider how stochastic effects studied above cause the noise-induced transitions from order to chaos.

## 2. Canard-type oscillations in the deterministic Rulkov model

Consider the two-dimensional neuron model

$$\begin{cases} x_{t+1} = \dfrac{\alpha}{1+{x_t}^2} + y_t \\ y_{t+1} = y_t - \sigma x_t - \beta \end{cases} \tag{1}$$

proposed by Rulkov [9]. Here, $x$ is the fast variable replicating the membrane potential, and $y$ is the slow variable. The system parameters $\alpha$, $\sigma$ and $\beta$ are positive. The parametric analysis of dynamic regimes of system (1) was carried out in [36].

In the present paper, we focus on the system dynamics in the parametric zone where the model (1) exhibits the Neimark–Sacker bifurcation [37] with the transition from the stable equilibrium to the quasiperiodic self-oscillations with the attractors in the form of the closed invariant curves. In system (1), such scenario will be observed for fixed $\sigma = \beta = 0.005$ under the variation of the parameter $\alpha$. Indeed, the Rulkov model (1) has a unique equilibrium $M(\bar{x}, \bar{y})$, where $\bar{x} = -1, \quad \bar{y} = -1 - \frac{\alpha}{2}$, and the point of the Neimark–Sacker bifurcation $\alpha_{NS} = 1.99$. For $0 < \alpha < 1.99$, the equilibrium $M$ is stable. When the increasing parameter $\alpha$ passes $\alpha_{NS}$, the equilibrium loses its stability, and a new attractor in the form of the the stable closed invariant curve (CIC) is born. In dependence of the rotation number $r$, these CICs can be formed by the quasiperiodic solutions ($r$ is irrational) or by the family of discrete cycles ($r$ is rational). In Fig. 1, extremal values of $x$-coordinates of attractors of system (1) are plotted by solid lines, and unstable equilibria are shown by dashed lines.

The most interesting feature of this deterministic model is that in the zone of CICs a phenomenon of the Canard explosion is observed. Indeed, in Fig. 1(b), one can see the narrow $\alpha$-subinterval where the amplitude of the oscillations sharply increases. Details of the changes of the form and size of CICs in this sub-interval are shown in the Fig. 2(a).

Corresponding changes in the stability level and frequency are illustrated in Fig. 2(b) and (c) by the Lyapunov exponents $\Lambda$ and the rotation number $r$. The rotation number is the average number of turnovers along the closed invariant curve per iteration. Here, the value $\alpha_* = 1.995278$ marks the epicenter of the Canard explosion. As one can see in Fig. 2(b), the function $\Lambda(\alpha)$ monotonously decreases, but close to $\alpha_*$, the Lyapunov exponent exhibits a sharp fall down. This means that the level of the stability of the Canard-type CICs substantially increases. As for the rotation number, the function $r(\alpha)$ demonstrates a qualitative transformation from the sharp decrease to the slow increase.