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Research paper

# Modeling and simulation of the hexagonal pattern formation of honeycombs by the immersed boundary method

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ABSTRACT

We present a simple mathematical model and numerical simulations of the hexagonal pattern formation of a honeycomb using the immersed boundary method. In our model, we assume that the cells have a circular shape at their inception and that there is a force acting upon the entire circumference of the cell. The net force from the individual cells is a key factor in their transformation from a circular shape to a rounded hexagonal shape. Numerical experiments using the proposed mathematical model confirm the hexagonal patterns observed in honeybee colonies.



## 1. Introduction

Honeybee nests are organized into parallel and vertically aligned self-synthesized wax combs. Both sides of the comb consist of arrays of hexagonal cells that house the brood and store resources. In a regular hexagonal cell, the six sides adjoin at a 120° degree angle [1]. The hexagonal structure appears in several studies in various research fields, such as pattern formation [2], honeycomb network [3], cationic liposome-DNA [4], and honeycomb formation [1,5]. The honeycomb is the most studied natural cellular structure [6]. Despite this, there is an ongoing debate about how these hexagonal cells are produced. For example, according to some sources, the comb structure is a result of a thermoplastic wax reaching a liquid equilibrium [7]. However, according to other researchers, the hexagonal structure is not produced via a liquid equilibrium process [1]. It was reported that the cells in a natural honeybee comb have a circular shape at inception but quickly transform into the well-known rounded hexagonal shape, see Fig. 1. Several studies on the honeycomb structure have been carried out [5,8].

In this study, we propose a simple mathematical model and perform numerical simulations of the hexagonal pattern formation of a honeycomb using the immersed boundary method [9]. The basic mechanism of our model as follows: First, we set the cells which have a circular shape at their birth. Second, we compute forces acting upon the entire circumference of the individual cells. Third, we calculate the net force from the individual cells. Fourth, we move the cell boundaries according to the net force. We repeat these last three steps until it reaches an equilibrium state or a specified time. Computational experiments of the proposed mathematical model demonstrate the hexagonal patterns observed in honeybee colonies.

The importance and advantage of the studied model is that we can generate complex hexagonal pattern using a simple mathematical model and has applications such as patient-specific 3D-printed cast, which will be described in more detail in Section 3.10.

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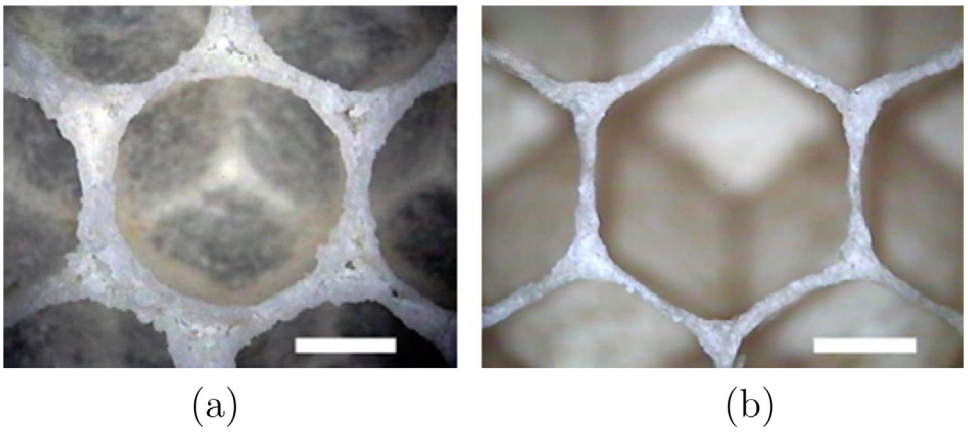


**Fig. 1.** Italian honeybee comb cell. (a) Inception and at (b) 2-day-old, scale bar is 2 mm. Reprinted from Karihaloo et al. [6] with permission from the Royal Society.

The novelty of the paper is to propose a mathematical model for a fundamental formation of hexagonal honeycomb structure and apply an efficient and accurate numerical method for solving the governing equation.

To the authors' knowledge this is the first attempt to model and simulate hexagonal pattern formation of the honeycomb structure in a large scale, including the interactions of cells. So far, most researches on hexagonal pattern formation focused on self-organized formation of hexagonal pattern through phase separation [10] or investigated the strength and stiffness under shear or compression [11–15].

The organization of this paper is as follows. In Section 2, we present the proposed mathematical model and numerical method for the hexagonal pattern formation. We present the simulation results in Section 3. In Section 4, conclusions are presented.

## 2. Mathematical model and numerical method

To model and simulate the hexagonal pattern formation of the honeycomb, we use the immersed boundary method (IBM) as a computational tool. This method was introduced by Peskin to study flow patterns around heart valves [9]. The IBM has been used by several researchers in various studies [16,17].

In the IBM, the fluid is represented on an Eulerian coordinate and the elastic structure is represented on a Lagrangian coordinate. The immersed elastic structures are typically represented as a collection of points. The velocity and pressure are defined on the Eulerian coordinate. The force exerted by the flexible structure on the fluid is considered as a source term in the momentum equation using a Dirac-delta function. Then, the Lagrangian points move with the fluid velocity interpolated through the Dirac-delta function. The governing equations are discretized by an Eulerian grid on the fluid and a Lagrangian grid on the immersed boundary. Approximations of the Dirac-delta function by a smoother function allow us to interpolate between the Eulerian and Lagrangian coordinates. Please refer to [18,19] and references therein for more details about the IBM.

Let $\mathbf{X}_k(s,t) = (\tilde{x}_k(s,t), \tilde{y}_k(s,t))$ be the immersed boundary for the $k$th cell at time $t$ for $1 \le k \le N_k$, where $0 \le s \le L_k(t)$ and $L_k(t)$ is the time-dependent length of the $k$-th boundary. Here, $N_k$ is the number of closed loops. Because the boundary is a closed curve, $\mathbf{X}_k(0,t) = \mathbf{X}_k(L_k(t),t)$. See Fig. 2.

We assume the cell boundaries move according to a net force on the boundaries and propose the following evolution equation:

$$\frac{\partial \mathbf{X}_k(s,t)}{\partial t} = \alpha \mathbf{F}(\mathbf{X}_k(s,t)), \tag{1}$$

where $\alpha$ is a proportional constant value and $\mathbf{F}$ is the net force resulting from the forces exerted outwardly by the individual cell boundaries, such as mechanical force or the heat flux caused by the honeybees [7]. Eq. (1) can be considered as a special case of the general Lagrange's equations of motion for $\mathbf{X}_k$ [20]. Lagrangian simulations have been studied in various fields [21,22].

Let a computational domain $\Omega = (0, L_x) \times (0, L_y)$ be partitioned in Cartesian geometry into a uniform mesh with mesh spacing $h$ as shown in Fig. 2. The center of each computational cell $\Omega_{ij}$ is located at $\mathbf{x}_{ij} = (x_i, y_j) = ((i-0.5)h, (j-0.5)h)$ for $i = 1, \ldots, N_x$ and $j = 1, \ldots, N_y$. Here, $N_x$ and $N_y$ are the numbers of cells in the $x$- and $y$-directions, respectively. We use $M$ Lagrangian points $\mathbf{X}^n_{k,l} = (\tilde{x}^n_{k,l}, \tilde{y}^n_{k,l})$ for $l = 1, \ldots, M$ at $t = n\Delta t$ to discretize the $k$th immersed boundary. Here, $\Delta t$ is the temporal step size. At $\mathbf{X}^n_{k,l}$, the corresponding outward unit normal vector $\mathbf{n}^n_{k,l}$ can be calculated by using three points $\mathbf{X}^n_{k,l-1}$, $\mathbf{X}^n_{k,l}$, $\mathbf{X}^n_{k,l+1}$ with the quadratic polynomial approximations [23]

$$\tilde{x}(t) = \alpha_1 t^2 + \beta_1 t + \gamma_1 \quad \text{and} \quad \tilde{y}(t) = \alpha_2 t^2 + \beta_2 t + \gamma_2.$$